Topics in Optimization

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A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Philosophy
in
Mathematics

The Chinese University of Hong Kong
July 2009

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摘要

向量優化中的一個重要問題是尋找集合的正真有效點在有效點中稠密的充分條件。 1953 年，Arrow, Barankin 和 Blackwell [1] 證明了歐幾裏德空間中一個這類的稠密性定理。特別地，他們證明了歐幾裏德空間中的一個緊凸集的正真有效點（在自然序關系意義下）在它的有效點中稠密。向量優化中的另一個重要問題是研究各類 Pareto 解的拓撲結構，例如連通性。本文將系統的介紹 Arrow, Barankin 和 Blackwell 的結果在 Banach（或賦範向量）空間中的幾個重要推廣。

Abstract

One important topic in vector optimization is to find sufficient conditions to guarantee that the efficient (minimal) points can be approximated by the special elements that can be scalarized, that is one seeks to ensure that the set of positive proper efficient points of a set is dense in the set of efficient points. In 1953, Arrow, Barankin and Blackwell [1] stated a celebrated density theorem of this kind in Euclidean spaces. In particular, they proved that the set of positive proper efficient points of a compact convex subset of a Euclidean space (under the natural order) is dense in the set of efficient points. Another important topic in vector optimization is the study of the topological structure such as the connectedness of various kinds of Pareto (minimal) solution sets. In this thesis, we study some important generalizations of Arrow, Barankin and Blackwell's results in the setting of Banach (or normed vector) spaces.

ACKNOWLEDGMENTS

I wish to express my gratitude to my supervisor Professor Kung Fu Ng and also to Professor Xi Yin Zheng for their guidance and for their many valuable suggestions.

I also want to express my appreciation to my friends: Fu Man Lau, Guoyin Li, Xing Li, Haixia Liang, Xiaoliu Wang, Wei Wang and Wei Yao who have been supportive while I work on this thesis.

My family has always encouraged me towards further education. I want to thank them for their encouragement and support.

Contents

Chapter 1

Introduction

In 1953, Arrow, Barankin and Blackwell [1] stated a celebrated density theorem in Euclidean spaces. They proved that the set of positive proper efficient points of a compact convex subset X_0 of a Euclidean space X is dense in the set of efficient points (if the Euclidean space is equipped with its natural ordering), that is for each "minimal" point x_0 in X_0 and for each $\varepsilon > 0$ there exists a "strictly positive" element x^* (with positive components) of X such that the corresponding linear functional restricting to X_0 attains its minimum at some point within the $\varepsilon-$neighbourhood of x_0. After that, many authors established some more general density results. Petschke [13] proved this density result in general normed spaces if A is weakly compact and the ordering cone is an abstract cone with a bounded base. Noting that a cone with a bounded base is a quasi-Bishop-Phelps cone, Ng and Zheng [12] generalized Petschke's result by proving the density result for a general normed space provided that the ordering cone is a quasi-Bishop-Phelps cone with a base. Without any compactness assumptions on the objective set, they also proved that if A is a closed convex complete set and the ordering cone is a closed convex cone with a weakly compact base, then the set of positive proper efficient points of A is dense in the set of efficient points of A. This study relates to the well-known Phelps Theorem, which stated that every weakly compact convex subset of a Banach space is the closed convex hull of its strongly exposed points. Since the completion of a normed space is a Banach space, the Phelps Theorem can easily be extended to the general normed spaces setting. Zheng, Yang and Teo [23] studied efficient point sets in general normed spaces where the ordering cone has a bounded base and established a parallel result as the Phelps Theorem. Theorem 3.2.4 gives a unified treatment of their results as well as the Phelps Theorem. Although the proof is basically modified from the literature (especially [23]), the presentation of Theorem 3.2.4 is new and encompasses the Phelps Theorem as well as the corresponding results in [23].

Another important theorem in [1] concerns the topological structure of Pareto solution sets and weak Pareto solution sets. Arrow, Barankin and Blackwell studied a vector optimization problem on a Euclidean space with its natural ordering. They proved that if the objective is a continuous affine mapping (a mapping is affine if it is a translation of a linear mapping) between two Euclidean spaces then Pareto solution sets and weak Pareto solution sets are the union of finitely many polyhedra and are pathwise connected. Zheng and Yang [22] generalized this theorem to piecewise linear case. They proved that if the objective is a cone-convex piecewise linear mapping (affine mappings are piecewise linear) between two general normed spaces, the constraint set is polyhedral, and the ordering cone is a polyhedral cone with nonempty interior, then the set of all weak Pareto solutions is the union of finitely many polyhedra and is pathwise connected. Very recently, in Banach spaces, Zheng [21] studied a vector optimization problem with a set-valued mapping as the objective whose graph is the union of finitely many polyhedra. He proved that if the ordering cone has nonempty interior and the objective is cone-convex or the ordering cone is polyhedral with nonempty interior, then the weak Pareto solution set and weak Pareto optimal value set are the union of finitely many polyhedra. Moreover, if the ordering cone has a weakly compact base, then the Pareto solution set and Pareto optimal value set are the union of finitely many polyhedra. Under the convexity assumptions of objective, he proved that the weak Pareto solution set is pathwise connected; moreover, the Pareto solution set and Pareto optimal value set are pathwise connected if the ordering cone has a weakly compact base. These are systematically surveyed in Chapter 4.

Chapter 2

Preliminary

2.1 Introduction

In this chapter, we introduce some notations and fundamental properties. They are mainly concerned with functional analysis and vector optimization.

2.2 Notations and fundamental properties

The following definitions are standard in vector optimization, see for instance [23, Chapter 1]. Throughout this subsection let X denote a (real) vector space (or a topological vector space if the topological consideration is needed).

Definition 2.2.1. *Let A be a subset of X. The set A is said to be convex, if for every x, $y \in A$,*

$$\lambda x + (1-\lambda)y \in A, \text{ for all } \lambda \in [0,1].$$

Remark 2.2.1.

(a) The intersection of arbitrarily many convex sets in X is convex.

(b) If S and T are nonempty convex subset of X, then $\alpha S + \beta T$ is convex for all $\alpha, \beta \in \mathbb{R}$. Consequently, for every $\bar{x} \in X$, the translated set $\bar{x} + S$ is convex

too. Here and throughout $\alpha S + \beta T$ denotes the algebraic operation of sets, that is, $\alpha S + \beta T := \{\alpha x + \beta y : \forall\, x \in S,\ \forall\, y \in T\}$.

For a subset S of a topological vector space X, the intersection of all convex subsets of X containing S is called the convex hull of S and is denoted by co(S). We use $\overline{\text{co}}(S)$ to denote the closure of the convex hull of S.

Definition 2.2.2. *Let C be a nonempty subset of X.*

(a) The set C is called a cone, if

$$x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C.$$

(b) A cone C is said to be pointed, if

$$C \cap (-C) = \{0\}.$$

(c) A cone C is called a convex cone if it is also convex.

(d) A nonempty convex subset Θ of a convex cone C is called a base of C, if

$$0 \notin \text{cl}(\Theta) \text{ and } \text{C} = \text{cone}\Theta := \{\text{t}\theta : \text{t} \geq \text{ and}\, \theta \in \Theta\}$$

where cl(Θ) *denotes the closure of* Θ.

Definition 2.2.3.

(a) A nonempty subset R of the product $X \times X$ is called a binary relation on X (we write xRy for $(x, y) \in R$).

(b) A binary relation $\preceq$ on X is called a partial ordering on X, if the following axioms are satisfied (for arbitrary $w, x, y, z \in X$):

1. $x \preceq x$;

2. $x \preceq y,\ y \preceq z \Rightarrow x \preceq z$;

3. $x \preceq y,\ w \preceq z \Rightarrow x + w \preceq y + z$;

4. $x \preceq y$, $\alpha \in \mathbb{R}_+ \Rightarrow \alpha x \preceq \alpha y$ *(where* $\mathbb{R}_+ = [0, +\infty)$*).*

(c) A partial ordering $\preceq$ *on* X *is said to be antisymmetric, if the following implication holds for arbitrary* $x, y \in X$*:*

$$x \preceq y,\; y \preceq x \Rightarrow x = y.$$

(d) A (real) vector space equipped with a partial ordering is called a partially ordered vector space.

It is important to note that in a partially ordered vector space two arbitrary elements may not be comparable. The relationship between a convex cone and a partial ordering in a real vector space is given by the following theorem, see for example [8, Theorem 1.18].

Theorem 2.2.1.

(a) If $\preceq$ *is a partial ordering on* X*, then the set*

$$C := \{x \in X \,|\, 0 \preceq x\}$$

is a convex cone. If, in addition, $\preceq$ *is antisymmetric, then* C *is pointed.*

(b) If C *is a convex cone in* X*, then the binary relation*

$$\preceq_C := \{(x, y) \in X \times X \,|\, y - x \in C\}$$

is a partial ordering on X*. If, in addition,* C *is pointed, then* $\preceq_C$ *is antisymmetric.*

A convex cone characterizing a partial ordering in a vector space is called an ordering cone. We denote $\preceq_C$ as a partial ordering induced by a convex cone C.

Definition 2.2.4. *Let* A *be a subset of* X *and* C *be a convex cone in* X*. We say that* $a \in A$ *is an efficient point of* A*, written as* $a \in E(A, C)$*, if*

$$x \in A \text{ and } x \preceq_C a \Rightarrow a \preceq_C x.$$

Clearly, if C is a pointed cone,

$$a \in E(A, C) \Leftrightarrow A \cap (a - C) = \{a\}.$$

Let A be a subset of a normed space X. The distance of $x \in X$ to A is defined by

$$d(x, A) := \inf\{\|x - a\| : a \in A\}.$$

Sometimes, we use $d_A(x)$ to denote $d(x, A)$ for convenience.

Definition 2.2.5. *(see [23]) Suppose that X is a normed space. Let C be a convex cone in X, and let A be a closed subset of X. A point $a \in A$ is a stable efficient point of A, written as $a \in \mathrm{Stab(A, C)}$, if the following implication holds:*

$$a_n \in A \text{ and } d(a_n, a - C) \to 0 \Rightarrow \|a_n - a\| \to 0. \tag{2.1}$$

Note that (2.1) entails

$$a' \in A,\ a' \in a - C \ \Rightarrow\ a' = a. \tag{2.2}$$

Thus,

$$\mathrm{Stab}(A, C) \subset E(A, C).$$

Recall that a function f from a normed space X to $\mathbb{R}$ is Lipschitz (of rank K) near $x \in X$, if there exist $K > 0$ and $\varepsilon > 0$ such that

$$|f(x_1) - f(x_2)| \le K\|x_1 - x_2\| \ \ \forall x_1, x_2 \in B_\varepsilon(x),$$

where $B_\varepsilon(x)$ denotes the open ball center at x with radius ε.

Definition 2.2.6. *(see [4]) (a) Let f be a function from a normed space X to $\mathbb{R}$. The generalized directional derivative of f at $x \in X$ in the direction $v \in X$, denoted by $f^\circ(x; v)$, is defined as follows:*

$$f^\circ(x; v) := \limsup_{y \to x\, t \downarrow 0} \frac{f(y + tv) - f(y)}{t}$$

(b) Let A be a nonempty closed set in X. The tangent cone of A at $a \in A$, denoted by $T_A(a)$, is the collection of such vector v: $d_A^\circ(x; v) = 0$.

(c) Let A be a nonempty closed set in X. The normal cone of A at $a \in A$ is defined by

$$N_A(a) = \{x^* \in X^* : \langle x^*, v\rangle \leq 0 \text{ for all } v \text{ in } T_A(a)\}.$$

Remark 2.2.2. *Equivalently (see [4, Theorem 2.4.5]),*

$$T_A(a) = \liminf_{x \xrightarrow{A} a,\, t \to 0^+} \frac{A - x}{t}$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Thus, $v \in T_A(a)$ if and only if for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all n.

The following definition is standard, see for instance [23] and [17].

Definition 2.2.7. *Let A be a closed and convex subset of a topological vector space X.*

(a) A point $a \in A$ is called an extreme point of A, written as $a \in \mathrm{Ext(A)}$, if

$$x_1, x_2 \in A,\ 0 < \lambda < 1 \text{ and } a = \lambda x_1 + (1 - \lambda)x_2 \Rightarrow a = x_1 = x_2. \tag{2.3}$$

(b) A nonempty set $S \subset A$ is called an extreme set of A if

$$x_1, x_2 \in A,\ 0 < \lambda < 1 \text{ and } \lambda x_1 + (1 - \lambda)x_2 \in S \Rightarrow x_1, x_2 \in S. \tag{2.4}$$

In particular, an extreme point $\bar{a}$ of A simply means that $\{\bar{a}\}$ is an extreme set of A.

(c) A point $a \in A$ is called a support point of A if there exists an $x^* \in X^*\backslash\{0\}$ such that

$$\langle x^*, a\rangle = \min\{\langle x^*, x\rangle : x \in A\} \tag{2.5}$$

where X^* denotes the topological dual space of X.

(d) If, in addition, X is a normed space, then a point $a \in A$ is called a strongly exposed point of A, written as $a \in \mathrm{Sexp}(A)$, if there exists $x^* \in X^* \backslash \{0\}$ such that (2.5) *holds and*

$$x_n \in A,\ \langle x^*, x_n \rangle \to \langle x^*, a \rangle \Rightarrow \|x_n - a\| \to 0 \tag{2.6}$$

In this case, we say that x^* strongly exposes A at a.

Note that

$$\mathrm{Sexp}(A) \subset \mathrm{Ext}(A). \tag{2.7}$$

Indeed, for any $a \in \mathrm{Sexp}(A)$, if $a = \lambda x_1 + (1-\lambda)x_2$ with $x_1,\ x_2 \in A$ and $0 < \lambda < 1$, then (2.5) implies that $\langle x^*, x_1 \rangle = \langle x^*, x_2 \rangle = \langle x^*, a \rangle$. This and (2.6) imply that $x_1 = x_2 = a$ and so $a \in \mathrm{Ext}(A)$. Hence, (2.7) holds.

Let C be a closed convex cone in a topological vector space X. Let C^+ denote the dual cone of C, that is,

$$C^+ := \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in C\}.$$

Let C^{+i} denote the set of all strictly positive continuous linear functionals of C, that is,

$$C^{+i} := \{x^* \in X^* : \langle x^*, x \rangle > 0 \text{ for all } x \in C \backslash \{0\}\}.$$

For any nonempty subset A of X, we define

$$\mathrm{Supp}(A, C^+) := \{a \in A : \text{there exists } x^* \in C^+ \backslash \{0\} \text{ such that (2.5) holds}\}, \tag{2.8}$$

$$\mathrm{Supp}(A, C^{+i}) := \{a \in A : \text{there exists } x^* \in C^{+i} \text{ such that (2.5) holds}\}, \tag{2.9}$$

$$\mathrm{Supp}_G(A, C^{+i}) := \{a \in A : C^{+i} \cap -N_A(a) \neq \emptyset\}. \tag{2.10}$$

Moreover,

$$\mathrm{Sexp}(A, C^{+i}) := \{a \in A : \text{there exists } x^* \in C^{+i} \text{ such that (2.5) and (2.6) holds}\}, \tag{2.11}$$

provided that X is a normed vector space. We say that a point in $\text{Supp}(A, C^+)$ is a positive support point of A and that a point in $\text{Sexp}(A, C^{+i})$ is a positive strongly exposed point of A. We say that a point in $\text{Supp}(A, C^{+i})$ is a positive proper efficient point of A and that a point in $\text{Supp}_G(A, C^{+i})$ is a generalized strictly positive support point of A. In some literature, $\text{Supp}(A, C^{+i})$ is written as $\text{Pos}(A, C)$. Note that

$$\text{Supp}(A, C^{+i}) \subset E(A, C). \tag{2.12}$$

Indeed, suppose $a \in \text{Supp}(A, C^{+i})$. Then there exists $x^* \in C^{+i}$ such that (2.5) holds. For any $a - c \in A$ with $c \in C$, $x^* \in C^{+i}$ implies that $\langle x^*, a - c\rangle \leq \langle x^*, a\rangle$. This and (2.5) imply that $\langle x^*, a - c\rangle = \langle x^*, a\rangle$. So $\langle x^*, c\rangle = 0$ and $c = 0$. Thus, $A \cap (a - C) = \{a\}$ and so $a \in E(A, C)$. Therefore, (2.12) holds.

2.3 Properties of polyhedra

In this section, let X and Y be normed spaces and let $C \subset Y$ be a convex cone with $\text{int}(C) \neq \emptyset$. If additional conditions are imposed, they will be explicitly specified.

Let $L(X, Y)$ denote the family of all continuous linear operators from X to Y; thus, $X^* = L(X, Y)$ with $Y = \mathbb{R}$. A subset P of X is called a polyhedron if there exist $x_1^*, \cdots, x_n^* \in X^*$ and $r_1, \cdots, r_n \in \mathbb{R}$ such that

$$P = \{x \in X : \langle x_i^*, x\rangle \leq r_i, i = 1, \cdots, n\}.$$

A nonempty subset F of a polyhedron P is called a face of P if F is an extremal subset. A polyhedron has finitely many faces (see [2] and [16] for instance).

Remark 2.3.1. *It follows from the definition that an intersection of finitely many polyhedra is a polyhedron.*

The following lemma (called Minkowski-Weyl Representation in some literature) is well known (see [2, Proposition 3.2.2] and [16, Theorem 19.1]).

Lemma 2.3.1. *Let P be a subset of a finite dimensional space Y. Then P is a polyhedron if and only if there exist $v_1, \cdots, v_p, v_{p+1}, \cdots, v_{p+q}$ in Y such that*

$$P = \{\Sigma_{i=1}^{p+q} t_i v_i : t_i \geq 0,\ i = 1, \cdots, p+q, \text{ and } \Sigma_{i=1}^{p} t_i = 1\}$$

The following Corollaries are immediate consequences of Lemma 2.3.1.

Corollary 2.3.2. *Let P_1, P_2 be polyhedra of a finite dimensional space Y. Then $P_1 + P_2$ is a polyhedra.*

Corollary 2.3.3. *Let Y_1 be a subspace of a finite dimensional space Y. Then Y_1 is a polyhedron.*

Theorem 2.3.4. *([22, Lemma 3.2]) Let P and Q be polyhedra in X and Y, respectively. Let $T \in L(X, Y)$ and suppose that Y is finite dimensional. Then $T(P)$ and $T^{-1}(Q)$ are polyhedra in Y and X, respectively.*

Proof. Take $x_1^*, \cdots, x_k^* \in X^*$ and $l_1, \cdots, l_k \in \mathbb{R}$ such that $P = \{x \in X : \langle x_i^*, x\rangle \leq l_i, i = 1, \cdots, k\}$. Let $X_1 = \bigcap_{i=1}^m \ker(x_i^*)$, where $\ker(x_i^*) = \{x \in X : \langle x_i^*, x\rangle = 0\}$. Then, there exists a finite dimensional subspace X_2 of X such that $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$. Let $P_2 = \{x \in X_2 : \langle x_i^*, x\rangle \leq l_i, i = 1, \cdots, k\}$. Then, P_2 is a polyhedron in X_2 and $P = X_1 + P_2$. By Lemma 2.3.1 there exist $h_1, \cdots, h_p, h_{p+1}, \cdots, h_{p+q} \in X_2$ such that

$$P_2 = \{\Sigma_{i=1}^{p+q} t_i h_i : t_i \geq 0, i = 1, \cdots, p+q \text{ and } \Sigma_{i=1}^{p} t_i = 1\}.$$

Hence

$$T(P) = \{\Sigma_{i=1}^{p+q} t_i T(h_i) : t_i \geq 0, i = 1, \cdots, p+q \text{ and } \Sigma_{i=1}^{p} t_i = 1\} + T(X_1).$$

Noting that $T(X_1)$ is a subspace of finite dimensional space Y, one has that $T(X_1)$ is a polyhedron by Corollary 2.3.3. This and Corollary 2.3.2 imply that $T(P)$ is a polyhedron in Y.

To prove that $T^{-1}(Q)$ is a polyhedron in X, take $y_1^*, \cdots, y_n^* \in Y^*$ and $r_1, \cdots, r_n \in \mathbb{R}$ such that $Q = \{y \in Y : \langle y_i^*, y\rangle \leq r_i, i = 1, \cdots, n\}$. Then,

$$\begin{aligned} T^{-1}(Q) &= \{x \in X : \langle y_i^*, T(x) \leq r_i, i = 1, \cdots, n\} \\ &= \{x \in X : \langle T^*(y_i^*), x\rangle \leq r_i, i = 1, \cdots, n\}, \end{aligned}$$

where T^* denotes the conjugate operator of T (so each $T^*(y_i^*) \in X^*$). Hence $T^{-1}(Q)$ is a polyhedron in X. □

Lemma 2.3.5. *Let X_1 and X_2 be closed subspaces of X with* $\dim(X_2) < +\infty$ *such that $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$. Then there exists $M > 0$ such that*

$$\|x_1\| + \|x_2\| \leq M\|x_1 + x_2\| \tag{2.13}$$

for all $x_1 \in X_1$ and for all $x_2 \in X_2$.

Proof. Suppose to the contrary that (2.13) does not hold. Then there exit $x_1^{(n)} \in X_1$ and $x_2^{(n)} \in X_2$ such that $\|x_1^{(n)}\| + \|x_2^{(n)}\| > n\|x_1^{(n)} + x_2^{(n)}\|$ for all $n \in \mathbb{N}$. Without loss of generality, we assume that $\|x_1^{(n)}\| + \|x_2^{(n)}\| = 1$ for all n. Then $\|x_1^{(n)} + x_2^{(n)}\| \to 0$. Since $\{x_2^{(n)}\}$ is a bounded sequence of a finite dimensional space X_2, there exists a convergent subsequence of $\{x_2^{(n)}\}$. With out loss of generality, we suppose that $x_2^{(n)} \to x_2^{(0)} \in X_2$. Then $x_1^{(n)} \to -x_2^{(0)} \in X_1$. Since $X_1 \cap X_2 = \{0\}$, one has $x_2^{(0)} = 0$, contradicting $\|x_1^{(n)}\| + \|x_2^{(n)}\| = 1$. □

Theorem 2.3.6. *[21, Lemma 2.1] A subset P of $X \times Y$ is a polyhedron if and only if there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_1 and Y_2 of Y and a polyhedron P_2 of $X_2 \times Y_2$ such that*

$$X \times Y = X_1 \times Y_1 + X_2 \times Y_2,\ (X_1 \times Y_1) \cap (X_2 \times Y_2) = \{(0,0)\}, \tag{2.14}$$

$$\max\{\dim(X_2), \dim(Y_2)\} < +\infty \text{ and } P = X_1 \times Y_1 + P_2. \tag{2.15}$$

Proof. Suppose that P is a polyhedron of $X \times Y$. Since $(X \times Y)^* = X^* \times Y^*$, there exist $x_i^* \in X^*, y_i^* \in Y^*$ and $r_i \in \mathbb{R}, i = 1, \cdots, m$, such that

$$P = \{(x, y) \in X \times Y : \langle x_i^*, x\rangle + \langle y_i^*, y\rangle \leq r_i, i = 1, \cdots, m\}.$$

Let

$X_1 = \{x \in X : \langle x_i^*, x\rangle = 0, i = 1, \cdots, m\}$ and $Y_1 = \{y \in Y : \langle y_i^*, y\rangle = 0, i = 1, \cdots, m\}$.

Then, X_1 and Y_1 are closed subspaces of X and Y of codimensions less or equal to m, respectively. Hence, there exist finite dimensional subspaces X_2 and Y_2 of X and Y, respectively, such that

$$X = X_1 + X_2,\ Y = Y_1 + Y_2,\ X_1 \cap X_2 = \{0\} \text{ and } Y_1 \cap Y_2 = \{0\},$$

(and so (2.14) holds). Let

$$P_2 = \{(x, y) \in X_2 \times Y_2 : \langle x_i^*, x\rangle + \langle y_i^*, y\rangle \le r_i, i = 1, \cdots, m\}.$$

It follows that $P = X_1 \times Y_1 + P_2$ and P_2 is a polyhedron in $X_2 \times Y_2$. Therefore, the necessity holds.

Conversely, suppose that there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_1 and Y_2 of Y and a polyhedron P_2 in $X_2 \times Y_2$ such that (2.14) and (2.15) hold. Take $z_i^* \in X_2^* \times Y_2^*$ and $r_i \in \mathbb{R}$, $i = 1, \cdots, k$, such that

$$P_2 = \{(x, y) \in X_2 \times Y_2 : \langle z_i^*, (x, y)\rangle \le r_i, i = 1, \cdots, k\}.$$

Since X_2 and Y_2 are finite dimensional, (2.14) and Lemma 2.3.5 imply that for any $(x, y) \in X \times Y$ there exists a unique element (x_1, x_2, y_1, y_2) of $X_1 \times X_2 \times Y_1 \times Y_2$ such that

$$(x, y) = (x_1, y_1) + (x_2, y_2) \text{ and } \|(x_1, y_1)\| + \|(x_2, y_2)\| \le M\|(x, y)\|,$$

where M is a constant independent on (x, y); let $Q(x, y) = (x_2, y_2)$. Then Q is a well-defined continuous linear operator from $X \times Y$ to $X_2 \times Y_2$. Let

$$\langle \tilde{z_i^*}, (x, y)\rangle = \langle z_i^*, Q(x, y)\rangle \text{ for all } (x, y) \in X \times Y.$$

Then $\tilde{z_i^*} \in (X \times Y)^*$ and

$$Q^{-1}(P_2) = \{(x, y) \in X \times Y : \langle \tilde{z_i^*}, (x, y)\rangle \le r_i, i = 1, \cdots, k\};$$

thus, $Q^{-1}(P_2)$ is a polyhedron in $X \times Y$. Noting that $X_1 + Y_1 + P_2 = Q^{-1}(P_2)$, it follows form (2.15) that P is a polyhedron of $X \times Y$. $\square$

Taking $Y = \{0\}$ in Theorem 2.3.6, we have the following result.

Corollary 2.3.7. *([21, Corollary 2.1]) A subset P of X is a polyhedron if and only if there exist closed subspaces X_1, X_2 of X and a polyhedron P_2 in X_2 such that*

$$X = X_1 + X_2,\ X_1 \cap X_2 = \{0\} \text{ and } \dim(X_2) < +\infty$$

and

$$P = X_1 + P_2.$$

The following two lemmas are taken from [21, Lemmas 2.2 and 2.3], and are the infinite dimensional extensions of the well known results in finite dimensional spaces, see Corollary 2.3.2.

Lemma 2.3.8. *Suppose that P_1 and P_2 are two polyhedra of X. Then $P_1 + P_2$ is a polyhedron and so is closed.*

Proof. By Corollary 2.3.7, there exist closed subspaces X_1, X_2 and $\bar{X}_1, \bar{X}_2$ of X and polyhedra $\hat{P}_1$ of X_2 and $\hat{P}_2$ of $\bar{X}_2$ such that

$$X = X_1 + X_2 = \bar{X}_1 + \bar{X}_2,\ X_1 \cap X_2 = \bar{X}_1 \cap \bar{X}_2 = \{0\} \text{ and } \max\{\dim(X_2), \dim(\bar{X}_2)\} < +\infty$$

and

$$P_1 = X_1 + \hat{P}_1 \text{ and } P_2 = \bar{X}_1 + \hat{P}_2.$$

Then

$$P_1 + P_2 = X_1 + \bar{X}_1 + \hat{P}_1 + \hat{P}_2. \tag{2.16}$$

By Corollary 2.3.2, $\hat{P}_1 + \hat{P}_2$ is a polyhedron of finite dimensional space $X_2 + \bar{X}_2$. This and Corollary 2.3.7 imply that $X_1 + \bar{X}_1 + \hat{P}_1 + \hat{P}_2$ is a polyhedron. By (2.16), we complete the proof. $\square$

Let A be a subset of $X \times Y$. We define

$$\Pi_X(A) = \{x \in X : \text{there exist } y \in Y \text{ such that } (x, y) \in A\},$$

and

$$\Pi_Y(A) = \{y \in Y : \text{there exist } x \in X \text{ such that } (x, y) \in A\}.$$

Lemma 2.3.9. *Let P be a polyhedron of $X \times Y$. Then $\Pi_X(P)$ is a polyhedron of X.*

Proof. By Theorem 2.3.6, there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_1 and Y_2 of Y and a polyhedron P_2 of $X_2 \times Y_2$ such that

$$\max\{\dim(X_2), \dim(Y_2)\} < +\infty \text{ and } P = X_1 \times Y_1 + P_2. \tag{2.17}$$

Hence

$$\Pi_X(P) = X_1 + \Pi_{X_2}(P_2). \tag{2.18}$$

By Lemma 2.3.1, there exist $(x_i, y_i) \in X_2 \times Y_2$ $(i = 1, \cdots, p+q)$ such that

$$P_2 = \{\Sigma_{i=1}^{p+q} t_i(x_i, y_i) : t_i \geq 0,\ i = 1, \cdots, p+q, \text{ and } \Sigma_{i=1}^{p} t_i = 1\}.$$

Hence

$$\Pi_{X_2}(P_2) = \{\Sigma_{i=1}^{p+q} t_i x_i : t_i \geq 0,\ i = 1, \cdots, p+q, \text{ and } \Sigma_{i=1}^{p} t_i = 1\}. \tag{2.19}$$

By Lemma 2.3.1,

$$\{\Sigma_{i=1}^{p+q} t_i x_i : t_i \geq 0,\ i = 1, \cdots, p+q, \text{ and } \Sigma_{i=1}^{p} t_i = 1\}$$

is a polyhedron in X_2 and so is $\Pi_{X_2}(P_2)$. This and Corollary 2.3.7 imply that $X_1 + \Pi_{X_2}(P_2)$ is a polyhedron. By (2.18) , we complete the proof. □

Remark 2.3.2. *Similar to the proof of Lemma 2.3.9, we can show that $\Pi_Y(P)$ is a polyhedron of Y if P is a polyhedron of $X \times Y$.*

Corollary 2.3.10. *[21, Corollary 2.2] Let $G : X \rightrightarrows Y$ be a set-valued mapping whose graph is a convex polyhedron of $X \times Y$. Let P and Q be polyhedra of X and Y, respectively. Then $G(P)$ and $G^{-1}(Q)$ are polyhedra of Y and X, respectively.*

Proof. Since P is a polyhedron in X, there exist $x_1^*, \cdots, x_k^* \in X^*$ and $c_1, \cdots, c_k \in \mathbb{R}$ such that

$$P = \{x \in X : \langle x_i^*, x\rangle \le c_i, \ \ i = 1, \cdots, k\}.$$

For each integer $i \in [1, k]$, define $z_i^* : X \times Y \to \mathbb{R}$ by $\langle z_i^*, (x, y)\rangle = \langle x_i^*, x\rangle$ for all $(x, y) \in X \times Y$. Then each $z_i^* \in (X \times Y)^*$. It follows from

$$P \times Y = \{(x, y) \in X \times Y : \langle z_i^*, (x, y)\rangle \le c_i, \ \ i = 1, \cdots, k\}$$

that $P \times Y$ is a polyhedron. Similarly, we can show that $X \times Q$ is a polyhedron. Noting that $\mathrm{Gr}(G)$ is a polyhedron, Remark 2.3.1 implies that $\mathrm{Gr}(G) \cap (P \times Y)$ and $\mathrm{Gr}(G) \cap (X \times Q)$ are polyhedra. By Lemma 2.3.9, $\Pi_Y(\mathrm{Gr}(G) \cap (P \times Y))$ and $\Pi_X(\mathrm{Gr}(G) \cap (X \times Q))$ are polyhedra of Y and X, respectively. Noting that $G(P) = \Pi_Y(\mathrm{Gr}(G) \cap (P \times Y))$ and $G^{-1}(Q) = \Pi_X(\mathrm{Gr}(G) \cap (X \times Q))$, we complete the proof. □

Lemma 2.3.11. *[22, Lemma 2.1] Suppose that P_i, $i = 1, \cdots, m$ are polyhedra of X. Let $I = \{1, \cdots, m\}$ and $I_0 = \{i \in I : \mathrm{int}(P_i) \neq \emptyset\}$. If $X = \bigcup_{i \in I} P_i$, then $X = \bigcup_{i \in I_0} P_i$.*

Proof. Let i be an arbitrary element in $I \backslash I_0$. Let $a \in X$ and $r \in (0, +\infty)$. Since $\mathrm{int}(P_i) = \emptyset$, one has $B(a, r) \not\subseteq P_i$, where $B(a, r)$ denotes the open ball with center a and radius r. Hence there exists $\tilde{a} \in B(a, r)$ such that $\tilde{a} \notin P_i$. This and the closedness of P_i imply that there exist $\tilde{r} \in (0, r - \|a - \tilde{a}\|)$ such that $B(\tilde{a}, \tilde{r}) \cap P_i = \emptyset$. It follows from the finiteness of $I \backslash I_0$ that for any $x \in X$ and $\varepsilon > 0$, there exist $\tilde{x} \in B(x, \varepsilon)$ and $\tilde{\varepsilon} \in (0, \varepsilon - \|x - \tilde{x}\|)$ such that

$$B(\tilde{x}, \tilde{\varepsilon}) \cap (\bigcup_{i \in I \backslash I_0} P_i) = \emptyset. \tag{2.20}$$

Suppose to the contrary that $X \neq \cup_{i\in I_0} P_i$. Since $\cup_{i\in I_0} P_i$ is closed, there exist $x_0 \in X$ and $\varepsilon_0 \in (0, +\infty)$ such that $B(x_0, \varepsilon_0) \cap (\cup_{i\in I_0} P_i) = \emptyset$. Take $\tilde{x} \in B(x_0, \varepsilon_0)$ and $\tilde{\varepsilon} \in (0, \varepsilon_0 - \|x_0 - \tilde{x}\|)$ such that (2.20) holds. Then,

$$B(\tilde{x}, \tilde{\varepsilon}) \cap (\bigcup_{i\in I} P_i) = \emptyset,$$

contradicting $X = \bigcup_{i\in I} P_i$. Therefore we must have $X = \bigcup_{i\in I_0} P_i$. □

A mapping $f : X \to Y$ is said to be piecewise linear if there exist polyhedra $P_1, \cdots, P_m$ in X, $\{T_1, \cdots, T_m\} \subset L(X, Y)$ and $\{b_1. \cdots, b_m\} \subset Y$ such that

$$X = \bigcup_{i=1}^{m} P_i \text{ and } f(x) = T_i(x) + b_i \ \forall x \in P_i \text{ and } 1 \leq i \leq m. \tag{2.21}$$

Hence, for all $i, j = 1, \cdots, m$,

$$T_i(x) + b_i = T_j(x) + b_j, \ \forall x \in P_i \cap P_j.$$

Let C be a convex cone in Y and let $\preceq_C$ be a partial ordering induced by C. We say that a mapping $f : X \to Y$ is $C-$convex if

$$f(tx_1 + (1-t)x_2) \preceq_C tf(x_1) + (1-t)f(x_2) \ \ \forall t \in [0, 1] \text{ and } x_1, x_2 \in X.$$

It is clear that f is $C-$convex if and only if $\text{epi}_C(f)$ is a convex subset of $X \times Y$, where

$$\text{epi}_C(f) := \{(x, y) : x \in X \text{ and } f(x) \preceq_C y\}$$

is the epigraph of f with respect to the ordering cone C.

Theorem 2.3.12. *[22, Lemma 2.2] Let f be a piecewise linear mapping defined by* (2.21)*. Then, f is $C-$convex if and only if*

$$T_i(x) + b_i \preceq_C f(x) \text{ for all } x \in X \text{ and } 1 \leq i \leq m. \tag{2.22}$$

Proof. Suppose that (2.22) holds. Let $\Omega_i = \{(x, y) \in X \times Y : T_i(x) + b_i \leq_C y\}$. Then, (2.22) implies that $\text{epi}_C(f) = \cap_{i=1}^m Q_i$. Thus, $\text{epi}_C(f)$ is convex and so f is

C-convex. Conversely, let $I_0 = \{i \in I : \text{int}(P_i) \neq \emptyset\}$. Then $X = \cup_{i \in I_0} P_i$. This and

$$T_i(x) + b_i = T_j(x) + b_j, \quad \text{for all}\, x \in P_i \cap P_j$$

imply that it is sufficient to prove that

$$T_i(x) + b_i \preceq_C f(x) \;\text{for all}\; x \in X \,\text{and}\, i \in I_0. \tag{2.23}$$

Let $x \in X$ and $i \in I_0$. Take $z \in \text{int}(P_i)$. Then there exists $t \in (0, 1)$ such that $z + t(x - z) \in P_i$, that is, $tx + (1 - t)z \in P_i$. It follows from the C-convexity of f that

$$T_i(tx+(1-t)z)+b_i = f(tx+(1-t)z) \preceq_C tf(x)+(1-t)f(z) = tf(x)+(1-t)(T_i(z)+b_i). \tag{2.24}$$

Since T_i is linear, it follows that (2.23) holds. $\square$

Chapter 3

Results on Efficient Point Sets

3.1 Introduction

One important problem in vector optimization theory is to find sufficient conditions to guarantee that the set of positive proper efficient points is dense in the set of efficient points. In 1953, Arrow, Barankin and Blackwell [1] stated a celebrated density theorem in a Euclidean space $\mathbb{R}^n$ (called Arrow-Barankin-Blackwell density theorem in some literature). They proved that the set of positive proper efficient points of a compact convex subset A of a Euclidean space $\mathbb{R}^n$ is dense in the set of efficient points of A if $\mathbb{R}^n$ is equipped with its natural ordering. Over the past decades, several authors have generalized the Arrow-Barankin-Blackwell density theorem in many different ways. In 1990, Petschke [13] proved this density result in general normed spaces if A is weakly compact and the ordering cone is an abstract cone with a bounded base. Note that a cone with a bounded base is a quasi-Bishop-Phelps cone (see page 34). In 2003, Ng and Zheng [12] studied density theorem in general normed spaces. They generalized Petschke's result and proved this density result if the ordering cone is a quasi-Bishop-Phelps cone with a base. Without any compactness assumptions on the objective set, Ng and Zheng [12] also established another important density theorem. They proved that if A is a closed convex complete set and the ordering cone is a closed convex cone with a weakly compact base, then the set of positive proper efficient points of A is dense in the set of efficient points of A. In 1999, Ferro [5] proved this density result in Banach spaces if A is a closed convex set and the ordering cone is a closed convex cone with a weakly compact base. Many authors studied the well-known Phelps theorem, which stated that every weakly compact convex subset of a Banach space is the closed convex hull of its strongly exposed points. Since the completion of a normed space is a Banach space, the Phelps theorem is still valid in general normed spaces. In 2007, Zheng, Yang and Teo [23] studied efficient point sets in general normed spaces where the ordering cone has a bounded base and established a parallel result as the Phelps theorem. They proved that if A is a weakly compact convex subset of a normed space, then the set of all efficient points of A is a subset of the closed convex hull of its positively strongly exposed points (see Theorem 3.2.4). Although the proof is basically modified from the literature (especially [23]), the presentation of Theorem 3.2.4 is new where we give a unification encompassing the Phelps theorem as well as the corresponding results in [23]. In this chapter, we will give a survey on some important generalizations of Arrow-Barankin-Blackwell density theorem.

3.2 Geometric results on efficient point sets

In this section, we present some characterization of the efficient point sets. The known Krein-Milman theorem in functional analysis tells us how to characterize a compact convex set in a topological vector space. In 1974, Phelps [14] got a sharper result under Banach space. Let X be a topological vector space, recall that the dual space X^* of X separates points of X if for any $x, y \in X$ with $x \neq y$, there exists $x^* \in X^*$ such that $\langle x^*, x\rangle \neq \langle x^*, y\rangle$.

Krein-Milman Theorem (see [17, Theorem 3.23]) Suppose X is a topological vector space such that X^* separates points of X. If A is a compact convex set in X, then A is the closed convex hull of its extreme points; that is,

$$A = \overline{\text{co}}(\text{Ext}(A)).$$

Phelps Theorem (see [6, p.215, Theorem 10]) Suppose A is a weakly compact convex subset of a Banach space X. Then A is the closed convex hull of its strongly exposed points; that is,

$$A = \overline{\text{co}}(\text{Sexp}(A)).$$

Moreover, the set of continuous linear functionals on X which strongly expose A is dense in X^*.

Remark 3.2.1. *Noting that the completion of a normed space is a Banach space, the Phelps Theorem still holds in a normed space.*

We recall the following separation theorem, which can be found in [17, Theorem 3.4].

Separation Theorem Suppose that A and B are disjoint nonempty convex sets in a topological vector space X.

1. If A is open, then there exist $x^* \in X^*$ and $r \in \mathbb{R}$ such that

$$\langle x^*, x\rangle < r \leq \langle x^*, y\rangle,$$

for every $x \in A$ and for every $y \in B$.

2. If A is compact, B is closed, and X is locally convex, then there exist $x^* \in X^*, r_1, r_2 \in \mathbb{R}$ such that

$$\langle x^*, x\rangle < r_1 < r_2 < \langle x^*, y\rangle,$$

for every $x \in A$ and for every $y \in B$.

The following result is also well known (see [17, Theorem 3.12] for instance).

Theorem 3.2.1. *Let A be a convex subset of a locally convex topological space X. Then the weak closure $\bar{A}_w$ of A is equal to its original closure $\bar{A}$.*

The following lemma is useful for us.

Lemma 3.2.2. *Let X be a topological vector space and suppose that X^* separates points of X. Let A be a weakly compact subset of X and let C be a weakly closed convex cone in X. Then,*

$$E(A,C) \cap \mathrm{Ext}(A) \neq \emptyset. \tag{3.1}$$

Moreover, if C is pointed, then for any subset Ω of A, the following equivalence holds:

$$E(A,C) \subset \Omega \Leftrightarrow E(A,C) \subset \Omega + C. \tag{3.2}$$

Proof. The assertion (3.1) is due to [3]. To establish (3.2), it is sufficient to prove the part "$\Leftarrow$". To do this, let $x \in E(A,C)$. Then there exist $\omega \in \Omega$ and $c \in C$ such that $x = \omega + c$. Hence $\omega = x$ and so $x \in \Omega$. □

The following lemma was presented in [23, Lemma 2.1] when X is a normed space and C is a closed convex pointed cone.

Lemma 3.2.3. *Let A be a weakly compact convex nonempty subset of a locally convex topological vector space X. Let C be a closed convex cone in X. Suppose that $x^* \in C^+\backslash\{0\}$, $\alpha := \min\{\langle x^*, x\rangle : x \in A\}$ and*

$$A_\alpha := \{a \in A : \langle x^*, a\rangle = \alpha\}.$$

Then, The following assertions holds:

(i) $A_\alpha \subset \mathrm{Supp}(A, C^+)$, and also $A_\alpha \subset \mathrm{Supp}(A, C^{+i})$ if $x^* \in C^{+i}$.

(ii) A_α is an extremal subset subset of A and, in particular $\mathrm{Ext}(A_\alpha) \subset \mathrm{Ext}(A)$.

(iii) $A \cap (a - C) \subset A_\alpha$ *whenever* $a \in A$, *and*

$$E(A_\alpha, C) \subset E(A, C) \tag{3.3}$$

(iv)

$$E(A_\alpha, C) \cap \mathrm{Ext}(A_\alpha) \neq \emptyset. \tag{3.4}$$

In particular, $x^*|_A$ *(the restriction of* x^* *to* A*) attains its minimum at some point of* $E(A, C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A, C^+)$ *(of* $E(A, C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A, C^{+i})$ *if* x^* *is assumed to belong to* C^{+i}*).*

Proof. Let $x \in A_\alpha$. Since $x^* \in C^+\backslash\{0\}$ and $\langle x^*, x\rangle = \min\{\langle x^*, a\rangle : a \in A\}$, one has $x \in \mathrm{Supp}(A, C^+)$. Hence $A_\alpha \subset \mathrm{Supp}(A, C^+)$. Similarly, one can prove that $A_\alpha \subset \mathrm{Supp}(A, C^{+i})$ if $x^* \in C^{+i}$. This complete the proof of (i).

To prove (ii), let $x \in A_\alpha$, $x_1, x_2 \in A$ and $\lambda \in (0, 1)$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. Then

$$\alpha = \langle x^*, x\rangle = \langle x^*, \lambda x_1 + (1 - \lambda)x_2\rangle = \lambda\langle x^*, x_1\rangle + (1 - \lambda)\langle x^*, x_2\rangle$$

It follows form the linearity of x^* and the definition of α that $\langle x^*, x_1\rangle = \langle x^*, x_2\rangle = \alpha$, which means that $x_1, x_2 \in A_\alpha$. Therefore, A_α is an extremal subset subset of A.

To prove (iii), let $a \in E(A_\alpha, C)$. Then $\langle x^*, a\rangle = \alpha$. This and $x^* \in C^+$ imply that

$$\langle x^*, x\rangle \leq \alpha, \quad \text{for all}\, x \in a - C.$$

It follows from the definition of α that

$$\langle x^*, x\rangle = \alpha \quad \text{for all}\, x \in A \cap (a - C).$$

Hence $A \cap (a - C) \subset A_\alpha$ and so $A \cap (a - C) = A_\alpha \cap (a - C)$. Let $x \in A$ and $x \preceq_C a$. Then $x \in A \cap (a - C) = A_\alpha \cap (a - C)$ and so $x \in A_\alpha$. Noting that $a \in E(A_\alpha, C)$, $x \in A_\alpha$ and $x \preceq_C a$ imply that $a \preceq_C x$. Hence $a \in E(A, C)$. This complete the proof of (iii).

Finally, we prove (iv). Since A is a nonempty weakly compact convex set, so is A_α. It follows from Lemma 3.2.2 that (3.4) holds. $\square$

The following (3.5) is taken from [23, Theorem 2.1].

Theorem 3.2.4. *Let C be a closed convex pointed cone in a locally convex topological vector space X and let A be a nonempty weakly compact convex subset of X. Then the following assertions are valid.*

(i) It holds that

$$E(A,C) \subset \overline{\mathrm{co}}(E(A,C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A,C^+)), \tag{3.5}$$

$$E(A,C) \subset \overline{\mathrm{co}}(E(A,C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A,C^+)) + C, \tag{3.6}$$

and that

$$A + C = \overline{\mathrm{co}}(E(A,C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A,C^+))) + C. \tag{3.7}$$

(ii) Suppose, in addition, that X is a normed space and that there exists a closed convex subset Θ of C such that

$$C \backslash \{0\} \subset \{\lambda\theta : \lambda > 0,\ \theta \in \Theta\} \tag{3.8}$$

and

$$0 < d_\Theta(0). \tag{3.9}$$

Then (3.5*), (3.6*) *and* (3.7*) *hold, where* (3.5*), (3.6*) *and* (3.7*) *are respectively same as* (3.5), (3.6) *and* (3.7) *but C^+ is replaced by C^{+i}.*

(iii) Suppose Θ in (ii) is bounded. Then

$$E(A,C) \subset \overline{\mathrm{co}}(\mathrm{Sexp}(A,C^{+i})), \tag{3.10}$$

$$E(A,C) \subset \overline{\mathrm{co}}(\mathrm{Sexp}(A,C^{+i})) + C \tag{3.11}$$

and

$$A + C = \overline{\mathrm{co}}(\mathrm{Sexp}(A,C^{+i})) + C. \tag{3.12}$$

Remark 3.2.2. *The set on the right-hand side of* (3.5), (3.5*) *and* (3.10) *are obviously contained in A. Moreover, if $C = \{0\}$, then $E(A, C) = A$, $C^+ = X^*$ and* $\mathrm{Ext}(A) \subset \mathrm{Supp}(A, C^+)$; *thus,* (3.5) *reduces to the Krein-Milman theorem.*

Remark 3.2.3. *If Θ in (ii) is empty, then $C = \{0\}$ (so $C^{+i} = X^* \backslash \{0\}$) and (iii) reduces to the generalization of the Phelps Theorem (note Remark 3.2.1). If Θ in (ii) is nonempty, then* (3.5*) *is the result of [23, Theorem 2.2].*

Remark 3.2.4. *A nonempty closed convex subset Θ of C satisfies* (3.8) *and* (3.9) *if and only if Θ is a base of C. In this case one can use the Separation Theorem (applied to $\{0\}$ and Θ) to get $e^* \in X^*$ of norm 1 and $\eta > 0$ such that*

$$\eta < \inf_{\theta \in \Theta} \{\langle e^*, \theta \rangle\} \tag{3.13}$$

(in particular $e^* \in C^{+i}$). If, in addition, Θ is assumed to be bounded, say $\|x\| \leq M < +\infty$ for all $\theta \in \Theta$, then

$$0 < -\|u^*\| \|\theta\| + \inf_{\theta \in \Theta} \{\langle e^*, \theta \rangle\} \leq \inf_{\theta \in \Theta} \{\langle u^* + e^*, \theta \rangle\}$$

for all $u^* \in B(0, \eta/M)$. Thus

$$B(e^*, \eta/M) \subset C^{+i}. \tag{3.14}$$

(3.10) *is the result of [23, Theorem 2.3] if $\Theta \neq \emptyset$.*

Remark 3.2.5. *It is easy to verify that* $\mathrm{Supp}(A, C^{+i}) \subset E(A, C)$. *Thus,* (3.5*), (3.6*) *and* (3.7*) *strengthen* (3.5), (3.6) *and* (3.7) *respectively. Similarly, it is easy to verify that* $\mathrm{Sexp}(A, C^{+i}) \subset E(A, C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A, C^{+i})$, *therefore* (3.10), (3.11) *and* (3.12) *strengthen* (3.5*), (3.6*) *and* (3.7*) *respectively.*

Proof. We first prove the following implications:

$$(3.5) \Leftrightarrow (3.6) \Rightarrow (3.7). \tag{3.15}$$

Suppose (3.6) holds, and denote

$$\Omega := \overline{\mathrm{co}}(E(A, C) \cap \mathrm{Ext}(A) \cap \mathrm{Supp}(A, C^+)). \tag{3.16}$$

Let $x \in E(A, C)$. Then, by (3.6), $x = \omega + c$ for some $\omega \in \Omega$ and $c \in C$. Then $\omega \in A$, $\omega \preceq_C x \in E(A, C)$. This implies that $\omega = x$ and $c = 0$. Therefore $x \in \Omega$ and so (3.5) holds. For (3.7), we only need to verify that

$$A + C \subset \Omega + C \tag{3.17}$$

(the inverse inclusion trivially holds as $A \supseteq \Omega$).

To do this, let $a \in A$. By Lemma 3.2.2, there exists some $y \in E(A \cap (a - C), C)$. Then $y \in E(A, C)$ (note that if $u \in A$ and $u \preceq_C y$ then $u \in y - C \subset (a - C) - C = a - C$ so $u \in A \cap (a - C)$ and hence $u = y$ as y is an efficient point of $A \cap (a - C)$). Since $y \in a - C$, it follows that $a \in y + C \subset E(A, C) + C$ and therefore $A \subset E(A, C) + C$ and (3.17) follows.We have just shown that (3.6) implies (3.5) and (3.7). Since it is trivial that (3.5) $\Rightarrow$ (3.6), (3.15) is shown. Similarly, one can show that (3.5*) $\Leftrightarrow$ (3.6*) $\Rightarrow$ 3.7*) and that (3.10) $\Leftrightarrow$ (3.11) $\Rightarrow$ (3.12). Thus, to complete the proof of (i), we need only show (3.6). Suppose to the contrary that there exists some $a \in E(A, C)$ but $a \notin \Omega + C$, where Ω is defined as in (3.16). Noting that $\Omega + C$ is convex and weakly closed (thanks to the given assumptions on A and C), one can apply the Separation Theorem to get some $x^* \in X^*$ of norm 1 such that

$$\langle x^*, a \rangle < \inf_{x \in \Omega + C} \{\langle x^*, x \rangle\}.$$

Since C is a cone, it follows that $\inf_{c \in C}\{\langle x^*, c \rangle\} = 0$ (so $x^* \in C^+$) and so

$$\langle x^*, a \rangle < \inf_{\omega \in \Omega} \{\langle x^*, \omega \rangle\}, \tag{3.18}$$

contradicting Lemma 3.2.3. This complete the proof of (i).

To prove (ii), we assume that $\Theta \neq \emptyset$ and proceed as in (i) but, instead of (3.16), let Ω be defined by

$$\Omega = \overline{\mathrm{co}}(\mathrm{Ext}(A) \cap \mathrm{Supp}(A, C^{+i})). \tag{3.19}$$

Suppose there exists $a \in E(A, C) \backslash (\Omega + C)$. Then, take $x^* \in X^*$ of norm 1 satisfying (3.18). Take $r \in \mathbb{R}$, $\delta > 0$ such that

$$\langle x^*, a\rangle < r < r + 4\delta < \inf_{\omega \in \Omega}\{\langle x^*, \omega\rangle\}. \tag{3.20}$$

Since A is weakly compact (so norm bounded, say with a bound $\zeta > 0$), pick $y^* \in C^{+i}$ with $\|y^*\| < \delta/\zeta$ (see (3.13)). Then $|\langle y^*, x\rangle| \leq \delta$ for each $x \in A$; in particular $|\langle y^*, a\rangle| \leq \delta$ and $|\langle y^*, \omega\rangle| \leq \delta$ for all $\omega \in \Omega$. Let $z^* := x^* + y^*$. then $z^* \in C^+ + C^{+i} \subset C^{+i}$ and

$$\langle z^*, a\rangle + 3\delta < r + 4\delta < \inf_{x \in \omega}\{\langle x^*, x\rangle\} \leq \inf_{x \in \Omega}\langle z^*, x\rangle + \delta.$$

Thus $\langle z^*, a\rangle < \inf_{x \in \Omega}\langle z^*, x\rangle$ and $a \in A$. This implies that the set of minimum points of $z^*|_A$ is disjoint from Ω, contradicting Lemma 3.2.3. This proves (2.8*) and complete the proof of (ii).

Finally, we prove (3.11) in (iii). We proceed as in (ii) but let Ω be defined by

$$\Omega := \overline{\text{co}}(\text{Sexp}(A, C^{+i})).$$

Let a, x^*, r, δ and ζ be as in the above proof for (iii). Since Θ is bounded, suppose $\|\theta\| \leq M < +\infty$ for all $\theta \in \Theta$. Pick $y_1^* \in C^{+i}$ and $\varepsilon \in (0, \frac{\delta}{2M})$ such that $B(y_1^*, \varepsilon) \subset C^{+i}$ (see (3.14)). By Remark 3.2.1, there exists $y_2^* \in B(y_1^*, \varepsilon)$ such that $x^* + y_2^*$ strongly expose A at some $z \in A$. Hence

$$\langle x^* + y_2^*, z\rangle = \min_{x \in A}\{\langle x^* + y_2^*, x\rangle\} \quad \text{and} \quad z \in \Omega. \tag{3.21}$$

It follows from $\|y_2^*\| < \frac{\delta}{M}$ and (3.20) that

$$\langle x^* + y_2^*, a\rangle + 2\delta < r + 4\delta < \inf_{x \in \Omega}\langle x^*, x\rangle < \inf_{x \in \Omega}\langle x^* + y_2^*, x\rangle + \delta.$$

Thus $\langle x^* + y_2^*, a\rangle < \inf_{x \in \Omega}\langle x^* + y_2^*, x\rangle$, contradicting (3.21). This complete the proof of (iii). □

Let A be a closed convex subset of X and $a \in A$. Recall [23], we say $a \in PC(A)$ if a net in A converges to a whenever it is weakly convergent to a. Moreover, the point a is a sequential PC point of A, written as $a \in PC_S(A)$, if a sequence in A converges to a whenever it is weakly convergent to a.

Proposition 3.2.5. *[23, Proposition 2.1] Let $a \in A$. Then the following statements are equivalent.*

(a) $a \in \mathrm{Sexp}(A, C^{+i})$.

(b) $a \in \mathrm{PC}_S(A)$ *and there exists an* $x^* \in C^{+i}$ *such that* $\langle x^*, a\rangle < \langle x^*, x\rangle$ *for any* $x \in A\backslash\{a\}$

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a). Since

$$x^* \in C^{+i} \,\text{and}\, \langle x^*, a\rangle < \langle x^*, x\rangle, \,\text{for any}\, x \in A\backslash\{a\},$$

we need only to prove that

$$x_n \in A \text{ with } \langle x^*, x_n\rangle \to \langle x^* a\rangle \Rightarrow \|x_n - a\| \to 0. \tag{3.22}$$

Suppose $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$. The weak compactness of A implies that $\{x_{n_k}\}$ has a weak convergent subsequence, say $\{x_{n_{k_l}}\}$ convergence to a in weak topology. $a \in \mathrm{PC}_S(A)$ implies that $\|x_{n_{k_l}} - a\| \to 0$ and so $\|x_n - a\| \to 0$. The statement (a) holds. □

Proposition 3.2.6. *[23, Proposition 3.2] Let A be a weak compact subset of normed space X, and $C \subset X$ be a closed convex cone. Then,* $\mathrm{PC}_S(A) \cap E(A, C) \subset \mathrm{Stab}(A, C)$. *In particular, if A is compact, then* $\mathrm{Stab}(A, C) = E(A, C)$.

Proof. Suppose $a \in \mathrm{PC}_S(A) \cap E(A, C)$ and $\{x_n\} \subset A$ with $d(x_n, a - C) \to 0$. Then, there exist $c_n \in C$ such that $\|x_n - (a - c_n)\| \to 0$. The weak compactness of A implies that, there exists an subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, satisfying $x_{n_k} \to x_0 \in A$ in weak topology. Hence, $c_{n_k} \to a - x_0 \in C$. It follows from $a \in E(A, C)$ that $a = x_0$ and so $x_{n_k} \to a \in A$ in weak topology. This and $a \in \mathrm{PC}_S(A)$ imply $\|x_{n_k} - a\| \to 0$. Therefore, $\|x_n - a\| \to 0$ and $a \in \mathrm{Stab}(A, C)$. If A is compact, then $\mathrm{PC}_S(A) = A$, and so $\mathrm{Stab}(A, C) = E(A, C)$. □

Proposition 3.2.7. *[23, Proposition 3.3] Let A be a weak compact subset of normed space X, and $C \subset X$ be a closed convex cone. Suppose that 0 is a sequential* PC *point of C. Then* $\operatorname{Stab}(A, C) = E(A, C)$.

Proof. It is obviously that $\operatorname{Stab}(A, C) \subset E(A, C)$. For any $a \in E(A, C)$ and $\{x_n\} \subset A$ with $d(x_n, a - C) \to 0$, there exists $c_n \in C$ such that $\|x_n - (a - c_n)\| \to 0$. The weak compactness of A implies that, there exists an subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, satisfying $x_{n_k} \to x_0 \in A$ in weak topology. Hence, $c_{n_k} \to a - x_0 \in C$. It follows from $a \in E(A, C)$ that $a = x_0$ and so $c_{n_k} \to 0$ in weak topology. This and $0 \in \mathrm{PC}_S(C)$ imply $\|c_{n_k}\| \to 0$. Therefore, $\|x_n - a\| \to 0$ and $a \in \operatorname{Stab}(A, C)$. □

3.3 Density of positive proper efficient point sets

To present the main theorem of this section, we need the following lemma, which can be found in [4, p.52, Corollary].

Lemma 3.3.1. *Let C be a closed subset of Banach space X and f be a function from X to $\mathbb{R}$. Suppose f attains a minimum over C at x and f is Lipschitz near x. Then*

$$0 \in \partial f(x) + N_C(x).$$

Theorem 3.3.2. *[23, Theorem 3.1] Let A be a weakly compact subset of normed space X. Suppose that C is a convex cone in X with a basis Θ. Then,*

$$\operatorname{Stab}(A, C) \subset \operatorname{cl}(\operatorname{Supp}_G(A, C^{+i})).$$

Proof. Suppose to the contrary that there exists $a \in \operatorname{Stab}(A, C)$ such that $a \notin \operatorname{cl}(\operatorname{Supp}_G(A, C^{+i}))$, then

$$d(a, \operatorname{Supp}_G(A, C^{+i})) > 0. \tag{3.23}$$

We claim that

$$d(a, \operatorname{Supp}_G(A, C^{+i}) + C) > 0. \tag{3.24}$$

In fact, if (3.24) does not hold, then there exists a sequence $\{x_n\}$ in $\mathrm{Supp}_G(A, C^{+i})$ such that

$$d(x_n, a - C) = d(a, x + C) \to 0.$$

Since $a \in \mathrm{Stab}(A, C)$ and $x_n \in \mathrm{Supp}_G(A, C^{+i}) \subset A$, $\|x_n - a\| \to 0$, contradicting (3.23). So (3.24) holds. Since Θ is a basis of C, $\delta := \inf\{\|\theta\| : \theta \in \Theta\} > 0$. Without loss of generality, we can assume $\delta > 1$. Take $c_0 \in C\backslash\{0\}$ such that

$$4\|c_0\| < d(a, \mathrm{Supp}_G(A, C^{+i}) + C), \tag{3.25}$$

and for any $n \in \mathbb{N}$, let $C_n := \mathrm{cl}(\mathrm{cone}(\Theta + \frac{1}{n}B_X)$. We claim that

$$A \cap (a - c_0 - C_n) \neq \emptyset, \text{ for all n large enough.} \tag{3.26}$$

In fact, if (3.26) does not hold, then $A \cap (a - c_0 - C_n) = \emptyset$ for all $n \in \mathbb{N}$. Hence there exists $x_n \in A$ such that $a - c_0 - x_n \in C_n$, that is, there exist $t_n \geq 0, \theta_n \in \Theta$ and $b_n \in B_X$ such that

$$\|a - c_0 - x_n - t_n(\theta_n + \frac{1}{n}b_n)\| \to 0. \tag{3.27}$$

By the weak compactness of A, we may assume $x_n \to x_0 \in A$ in weak topology (passing to a subsequence if necessary). This and $\|\theta_n + \frac{1}{n}b_n\| > \delta - 1 > 0$ imply that $\{t_n\}$ is a bounded sequence and so $\frac{t_n}{n}b_n \to 0$. By (3.27), one has $t_n\theta_n \to a - c_0 - x_0 \in C$ in weak topology and so $x_0 \in a - c_0 - C \subset a - C$. It follows from $a \in \mathrm{Stab}(A, C) \subset E(A, C)$ that $x_0 = a$ and so $t_n\theta_n \to a - a_0 - x_0 = -c_0 \in C$. This contradicts to $c_0 \in C\backslash\{0\}$. Hence (3.26) holds. Let X_0 denote the completion space of X. Then $X_0^* = X^*$ and so A is a weakly compact subset of Banach space X_0. Let $f(x) := d(x, a - c_0 - C_n)$, for all $x \in X_0$. Then f is a convex continuous function on X_0 and so is weakly lower semicontinuous. It follows from the weak compactness of A, there exists $a_n \in A$ such that

$$f(a_n) = \min\{f(x) : x \in A\}. \tag{3.28}$$

By Lemma 3.3.1,

$$0 \in \partial f(a_n) + N(A, a_n). \tag{3.29}$$

For any $x^* \in \partial f(a_n)$, noting that f is convex, one has

$$\langle x^*, x - a_n \rangle \leq f(x) - f(a_n), \quad \forall x \in X_0.$$

Hence for any $c \in C$,

$$\langle x^*, a - c_0 - c - a_n \rangle \leq f(a - c_0 - c) - f(a_n) = -d(a_n, a - c_0 - C_n) < 0. \tag{3.30}$$

(The last inequality holds by (3.26) and $a_n \in A$.)

Since C_n is a cone, (3.30) implies $x^* \in C_n^+ \backslash \{0\}$. Hence, for any $\theta \in \Theta$,

$$\langle x^*, \theta - \frac{\|x^*\|}{n} \rangle = \inf\{\langle x^*, \theta + \frac{1}{n} b \rangle : b \in B_X\} \geq 0,$$

and so $\langle x^*, \theta \rangle > 0$ and $x^* \in C^{+i}$. The arbitrary of x^* implies $\partial f(a_n) \subset C^{+i}$ and so $a_n \in \mathrm{Supp}_G(A, C^{+i})$. This and (3.29) imply $a_n \in \mathrm{Supp}_G(A, C^{+i})$. By (3.28), we have

$$d(a_n, a - c_0 - C_n) = f(a_n) \leq f(a) \leq \|c_0\|.$$

Hence, there exist $\tilde{t}_n \geq 0$, $\tilde{\theta}_n \in \Theta$ and $\tilde{b}_n \in B_X$ such that

$$\|a_n - (a - c_0 - \tilde{t}_n(\tilde{\theta}_n + \frac{1}{n}\tilde{b}_n))\| \leq 2\|c_0\|.$$

The weak compactness of A and $\|\tilde{\theta}_n + \frac{1}{n}\tilde{b}_n\| \geq \delta - 1 > 0$ imply that $\{\tilde{t}_n\}$ is a bounded sequence and

$$\|a - \tilde{t}_n\tilde{\theta}_n - a_n\| \leq 3\|c_0\| + \frac{\tilde{t}_n}{n}.$$

Noting that $a_n \in \mathrm{Supp}_G(A, C^{+i})$,

$$d(a, \mathrm{Supp}_G(A, C^{+i}) + C) \leq 3\|c_0\| + \frac{\tilde{t}_n}{n}.$$

Letting $n \to \infty$, by the boundedness of $\{\tilde{t}_n\}$ and (3.25),

$$d(a, \mathrm{Supp}_G(A, C^{+i}) + C) \leq 3\|c_0\| < \frac{3}{4} d(a, \mathrm{Supp}_G(A, C^{+i}) + C),$$

contradiction. □

Recall [12] that a closed convex cone C in a normed space X is called a quasi-Bishop-Phelps cone if there exists a compact subset G of X^* such that

$$C \subset \{x \in X : \|x\| \leq \max\{\langle x^*, x\rangle : x^* \in G\}\}. \tag{3.31}$$

Remark 3.3.1. *The following implications are known:*
C has a bounded base $\Rightarrow$ C is a quasi-Bishop-Phelps cone $\Rightarrow$ $0 \in \mathrm{PC}_S(C)$.

Indeed, let Θ be a bounded base of C. Then there exist $r_1, r_2, M \in (0, \infty)$ and $x^* \in X^*$ such that

$$r_1 < \langle x^*, x\rangle < r_2, \text{ for any } x \in \Theta,$$

and

$$\|x\| \leq M, \text{ for any } x \in \Theta.$$

So,

$$\frac{\langle x^*, x\rangle}{\|x\|} \geq \frac{r_1}{\|x\|} \geq \frac{r_1}{M}, \text{ for any } x \in \Theta.$$

Let $f = \frac{M}{r_1}x^*$. Then

$$C \subset \{x \in X : \|x\| \leq f(x)\},$$

that is, C is a quasi-Bishop-Phelps cone.

If C is a quasi-Bishop-Phelps cone, then there exists a compact subset G of X^* such that

$$C \subset \{x \in X : \|x\| \leq \sup\{\langle x^*, x\rangle : x^* \in G\}\}.$$

The compactness of G implies that there exist $x_1^*, x_2^*, \cdots, x_n^* \in G$ such that

$$G \subset \bigcup_{i=1}^{n}(x_i^* + \frac{1}{2}B_X^*). \tag{3.32}$$

Let $x \in C\backslash\{0\}$. Then, (3.31) and (3.32) imply that

$$\|x\| \leq \max\{\langle x^*, x\rangle : i = 1, \cdots, n\} + \frac{\|x\|}{2},$$

and so

$$\|x\| \leq \max\{\langle 2x^*, x\rangle : i = 1, \cdots, n\}.$$

Thus, $0 \in \mathrm{PC}_S(C)$.

Corollary 3.3.3. *[12, Theorem 4.2] Let A be a weakly compact convex set in a normed space X, and let $C \subset X$ be a closed convex cone with a base. Then,*

$$E(A, C) \subset \mathrm{cl}(\mathrm{Supp}(A, C^{+i})) \tag{3.33}$$

if C is a quasi-Bishop-Phelps cone.

Proof. Since A is convex, $\mathrm{Supp}_G(A, C^{+i}) = \mathrm{Supp}(A, C^{+i})$. It follows from Remark 3.3.1, Proposition 3.2.7 and Theorem 3.3.2 that (3.33) holds. □

Let C be a closed convex cone of a normed space X. Recall [20] that a nonempty set $A \subset X$ has the domonation property with respect to C if for each $x \in A$, there exists $\bar{x} \in E(A, C)$ such that $\bar{x} \leq_C x$. If the convex cone C has a base Θ, let $\mathrm{int}_\Theta(C^+) = \{f \in C^+ \mid \inf\{f(\theta) \mid \theta \in \Theta\} > 0\}$ for convenience.

Lemma 3.3.4. *[20, Lemma 3.1] Let X be a locally convex topological space, and $C \subset X$ a convex cone with a bounded base Θ. Then,*

(a) If $x_{n+1} \preceq_C x_n$ for all $n \in \mathbb{N}$, and there exists $f \in \mathrm{int}_\Theta(C^+)$ such that the scalar sequence $\{f(x_n)\}$ is lower bounded, then $\{x_n\}$ is a Cauchy sequence.

(b) If $\{x_n\} \subset C$ and there exists $f \in \mathrm{int}_\Theta(C^+)$ such that $f(x_n) \to 0$, then $x_n \to 0$.

Proof. (a) Let $\alpha := \inf\{f(\theta) \mid \theta \in \Theta\}$. Then $\alpha > 0$. Since Θ is bounded, for each neighborhood V of 0, there exists $\delta > 0$ such that

$$t\Theta \subset V \quad \text{for all } t \in [0, \delta]. \tag{3.34}$$

It is easy to verify that $\{f(x_n)\}$ is a nonincreasing bounded sequence and so is convergent. Hence, there exists $n_0 \in \mathbb{N}$ such that $f(x_n - x_m) \leq \alpha\delta$ for all $m \geq n \geq n_0$. Noting that $x_n - x_m \in C$ for all $m \geq n \geq n_0$, there exists $\lambda_{nm} \geq 0$ and $\theta_{nm} \in \Theta$ such that $x_n - x_m = \lambda_{nm}\theta_{nm}$. Thus,

$$\alpha\delta > f(x_n - x_m) = \lambda_{nm} f(\theta_{nm}) \geq \lambda_{nm}\alpha,$$

and so $\lambda_{nm} < \delta$ for all $m \geq n \geq n_0$. By , whenever $m \geq n \geq n_0$, $x_n - x_m = \lambda_{nm}\theta_{nm} \in V$. Therefore, $\{x_n\}$ is a Cauchy sequence.

(b)Since $x_n \in C$, there exist $\lambda \geq 0$ and $\theta_n \in \Theta$ such that $x_n = \lambda_n \theta_n$. Hence,

$$0 \leq \lambda_n \alpha \leq \lambda_n f(\theta_n) = f(x_n).$$

This and $f(x_n) \to 0$ imply that $\lambda_n \to 0$. Noting that Θ is bounded, one has $x_n = \lambda_n \theta_n \to 0$. □

To present a main result of this section, we need the following theorem, which is a central result in [20].

Theorem 3.3.5. *Let X be a locally convex topological vector space, $C \subset X$ a closed convex cone, and $A \subset X$ a sequentially complete set. Suppose that C has a bounded base Θ and there exists $f \in \mathrm{int}_\Theta(C^+)$ such that f is lower bounded on A. Then A has the domination property with respect to C. Consequently, $E(A, C) \neq \emptyset$.*

Proof. For any $x_0 \in A$,

$$-\infty < \inf\{f(x) \mid x \in A\} \leq \inf\{f(x) \mid x \in A \cap (x_0 - C)\}.$$

Hence, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A \cap (x_0 - C)$ such that

$$f(x_\varepsilon) < \inf\{f(x) \mid x \in A \cap (x_0 - C)\} + \varepsilon. \tag{3.35}$$

It is easy to verify that

$$x_\varepsilon \preceq_C x_0. \tag{3.36}$$

Hence, by (3.35) and (3.36), we can choose a sequence $\{x_n\} \subset A$ such that

$$x_n \preceq_C x_{n-1} \tag{3.37}$$

and

$$f(x_n) < \inf\{f(x) \mid x \in A \cap (x_{n-1} - C)\} + \frac{1}{n}, \quad \text{for all } n \in \mathbb{N}. \tag{3.38}$$

By (a) of Lemma 3.3.4, $\{x_n\}$ is a Cauchy sequence in A. Since A is sequentially complete, there exists $\bar{x} \in A$ such that $x_n \to \bar{x}$. For each $n_0 \in \mathbb{N}$, we have $x_{n_0} - x_n \in C$ for all $n \geq n_0$. The closedness of C implies that $x_{n_0} - x_n \to x_{n_0} - \bar{x} \in C$. Hence $\bar{x} \leq_C x_n$ for all $n \in \mathbb{N}$. We claim that $\bar{x} \in E(A, C)$. Otherwise, there exists $\bar{y} \in A$ such that $\bar{x} - \bar{y} \in C$ and $\bar{x} \neq \bar{y}$. By $f \in \mathrm{int}_\Theta(C^+)$ and $\bar{x} \leq_C x_n$ for all $n \in \mathbb{N}$, one has $f(\bar{x} - \bar{y}) > 0$ and $\bar{y} \leq_C x_n$ for all n. Hence $f(\bar{x}) > f(\bar{y})$ and $\bar{y} \in A \cap (x_n - C)$. By (3.38),

$$f(x_n) - \frac{1}{n} < \inf\{f(x) \,|\, x \in A \cap (x_{n-1} - C)\} \leq f(\bar{y}).$$

Letting $n \to +\infty$ one has $f(\bar{x}) \leq f(\bar{y})$, which contradicts to $f(\bar{x}) > f(\bar{y})$. □

Lemma 3.3.6. *[9, Section 3.8.3] Let X be a Hausdorff topological vector space and let C be a cone in X with a bounded closed base. Then C is closed.*

Lemma 3.3.7. *[12, Lemma 4.1] Let C and K be closed convex pointed cones of a normed space X such that $C\backslash\{0\} \subset \mathrm{int}(K)$, and let A be a closed convex subset of X. Then $E(A, K) \subset \mathrm{Supp}(A, C^{+i})$.*

Proof. Let $x \in E(A, K)$. Then, $(x - K) \cap A = \{x\}$ and so $(x - \mathrm{int}(K)) \cap A = \emptyset$. By the Separation Theorem, there exists $f \in X^*\backslash\{0\}$ such that

$$f(x - y) \leq f(z) \ \ \forall\, y \in \mathrm{int}(K) \text{ and } \forall\, z \in A..$$

This implies that $f(y) > 0$ for all $y \in \mathrm{int}(K)$ and $f(x) \leq f(z)$ for all $z \in A$. Noting that $C\backslash\{0\} \subset \mathrm{int}(K)$, one has $f \in C^{+i}$ and so $x \in \mathrm{Supp}(A, C^{+i})$. Hence, $E(A, K) \subset \mathrm{Supp}(A, C^{+i})$. □

Theorem 3.3.8. *[12, Theorem 4.3] Let A be a closed convex subset of a normed space X and let $C \subset X$ be a closed convex cone with a weakly compact base Θ. Suppose $x \in E(A, C)$. If there exists $r > 0$ such that $A \cap B(x, r)$ is complete, then $x \in \mathrm{cl}(\mathrm{Supp}(A, C^{+i}))$, where $B(x, r)$ denotes the closed ball of X with center x and radius r. Consequently, $E(A, C) \subset \mathrm{cl}(\mathrm{Supp}(A, C^{+i}))$ if A itself is complete.*

Proof. Since Θ is a base, without loss of generality, we can assume that

$$\inf\{\|\theta\| \mid \theta \in \Theta\} > 1$$

Let $A_r := A \cap B(x, r)$ and let $C_n := \text{cone}(\Theta + \frac{1}{n}B_X)$ for each $n \in \mathbb{N}$, where B_X denotes the closed unit ball of X. The convexity of $\Theta + \frac{1}{n}B_X$ implies that C_n is convex. Then, by Lemma 3.3.6, each C_n is a closed convex cone with a bounded closed base $\Theta + \frac{1}{n}B_X$. This and the completeness of A_r imply that $A_r \cap (x - C_n)$ is complete. By Theorem 3.3.5, $E(A_r \cap (x - C_n), C_n) \neq \emptyset$. For each n, take an $a_n \in E(A_r \cap (x - C_n), C_n)$. Then, there exist $\theta_n \in \Theta$, $b_n \in B_X$ and $t_n \geq 0$ such that

$$a_n = x - t_n(\theta_n + \frac{b_n}{n}). \tag{3.39}$$

Since A_r and $\Theta + \frac{1}{n}B_X$ are bounded, $\{t_n\}$ is a bounded sequence. Noting that Θ is a weakly compact set, without loss of generality, we can assume that

$$t_n \to t \geq 0 \text{ and } \theta_n \to \theta \in \Theta \text{ in weak topology}. \tag{3.40}$$

This and (3.3) imply that

$$x - t_n\theta_n \to x - t\theta \text{ and } a_n \to x - t\theta \text{ in weak topology}. \tag{3.41}$$

Since A and $x - C$ are weakly closed, $x - t\theta \in A \cap (x - C)$. Noting that $x \in E(A, C)$, one has that $t = 0$. Hence $a_n \to x$. Therefore, it is sufficient to show that $a_n \in \text{Supp}(A, C^{+i})$ for sufficiently large n. Without loss of generality, we can assume that $\|a_n - x\| < \frac{r}{2}$ for each n. Hence $A \cap B(a_n, \frac{r}{2}) \subset A_r$. By Lemma 3.3.7, $a_n \in E(A_r \cap (x - C_n), C_n) \subset E(A_r, C_n) \subset \text{Supp}(A_r, C^{+i})$. Hence, there exists $f_n \in C^{+i}$ such that

$$f_n(a_n) = \min\{f_n(x) \mid x \in A_r\}. \tag{3.42}$$

On the other hand, if $z \in A \backslash A_r$, $\|z - a_n\| > \frac{r}{2}$. This and the convexity of A imply that

$$a_n + \frac{r}{2\|z - a_n\|}(z - a_n) \in A \cap B(a_n, \frac{r}{2}) \subset A_r.$$

By (3.42), $f_n(a_n) \leq f_n(a_n + \frac{r}{2\|z-a_n\|}(z - a_n))$ and so $f_n(a_n) \leq f_n(z)$. Therefore, $f_n(a_n) \leq f_n(y)$ for all $y \in A$ and so $a_n \in \mathrm{Supp}(A, C^{+i})$. □

The following corollary, which is a main result in [5], is an immediate consequence of Theorem 3.3.8.

Corollary 3.3.9. *[5, Theorem 3.1 (iii)] Let A be a closed convex subset of a Banach space X and let C be a closed convex cone of X having a weakly compact base. Then,*

$$\mathrm{Supp}(A, C^{+i}) \subset E(A, C) \subset \mathrm{cl}(\mathrm{Supp}(A, C^{+i})).$$

Chapter 4

Pareto Solutions of Polyhedral-valued Vector Optimization

4.1 Introduction

In 1953, Arrow, Barankin and Blackwell [1] studied a vector optimization problem on a Euclidean space with its natural ordering. They proved that if the objective is a continuous affine mapping (a mapping is affine if it is a translation of a linear mapping) between two Euclidean spaces then the Pareto solution sets and the weak Pareto solution sets are the union of finitely many polyhedra and are pathwise connected. In 2008, Zheng and Yang [22] generalized this theorem to piecewise linear case. They proved that if the objective is a cone-convex piecewise linear mapping (affine mappings are piecewise linear) between two general normed spaces, the constraint set is polyhedral, and the ordering cone is a polyhedral cone with nonempty interior, then the set of all weak Pareto solutions is the union of finitely many polyhedra and is pathwise connected. Very recently, in Banach spaces, Zheng [21] studied a vector optimization problem with a set-valued mapping as the objective whose graph is the union of finitely many polyhedra. He proved that if the objective is cone-convex (the interior of the ordering cone is nonempty) or the ordering cone is polyhedral with nonempty interior, then the weak Pareto solution set and weak Pareto optimal value set are the union of finitely many polyhedra. Moreover, if the ordering cone has a weakly compact base, then the Pareto solution set and Pareto optimal value set are the union of finitely many polyhedra. Under the convexity assumptions of objective, he proved that the weak Pareto solution set is pathwise connected; moreover, the Pareto solution set and Pareto optimal value set are pathwise connected if the ordering cone has a weakly compact base. In this chapter, we will give a survey of some results on structure and connectedness of the weak Pareto solution set, Pareto solution set, weak Pareto optimal value set and Pareto optimal value set.

4.2 The structure of weak Pareto solution sets

In this section, let X, Y be normed spaces and let $C \subset Y$ be a closed convex cone with nonempty interior, which specifies a preorder $\preceq_C$ in Y: for $y_1, y_2 \in Y$, $y_1 \preceq_C y_2 \Leftrightarrow y_2 - y_1 \in C$. By $y_1 \prec_C y_2$, we mean that $y_2 - y_1 \in \mathrm{int}(C)$. If additional conditions are imposed, they will be explicitly specified.

For a subset A of Y, recall that $a \in A$ is a weak Pareto efficient point of A if there is no element $y \in A$ such that $y \prec_C a$. We denote by $WE(A, C)$ the set of all weak Pareto efficient points of A. It is clear that

$$a \in WE(A, C) \Leftrightarrow a \in A \text{ and } (a - \mathrm{int}(C)) \cap A = \emptyset.$$

Recall that a set A of a topological space is said to be path (pathwise) connected if, for every two points $x, y \in A$, there exists a continuous function $\phi : [0, 1] \to A$ such that $\phi(0) = x$ and $\phi(1) = y$. Let $F : X \rightrightarrows Y$ be a set-valued mapping, $a_1^*, \cdots, , a_n^* \in X^*$ and $r_1, \cdots, r_n \in \mathbb{R}$. Consider the following set-valued vector optimization problem:

$$C - \min F(x) \text{ subject to } \langle a_j^*, x\rangle \le r_i, j = 1, \cdots, n. \tag{4.1}$$

For convenience, let Γ denote the feasible set of (4.1), that is,

$$\Gamma := \{x \in X : \langle a_j^*, x\rangle \le r_j, j = 1, \cdots, n\}.$$

A vector $\bar{x} \in \Gamma$ is called a Pareto (resp. weak Pareto) solution of (4.1) if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in E(F(\Gamma), C)$ (resp. $\bar{y} \in WE(F(\Gamma), C)$); in this case, $\bar{y}$ is called a Pareto (resp. weak Pareto) optimal value of (4.1). We denote by S (resp. S_w) and V (resp. V_w) the set of all Pareto (resp. weak Pareto) solutions of (4.1) and the set of all Pareto (resp. weak Pareto) optimal values of (4.1), respectively. It is clear that

$$S = \Gamma \cap F^{-1}(V) \text{ and } S_w = \Gamma \cap F^{-1}(V_w), \tag{4.2}$$

where $F^{-1}(V) = \{x \in X : \text{there exists } y \in V \text{ such that } y \in F(x)\}$.

The following equation is useful for us.

$$S_w = \Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w)). \tag{4.3}$$

Indeed, let $x \in S_w$. Then $x \in \Gamma$ and there exists $y \in F(x)$ such that $y \in WE(F(\Gamma), C) = V_w$. Thus, $(x, y) \in \mathrm{Gr}(F) \cap (\Gamma \times V_w)$ and so $x \in \Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w))$. Hence $S_w \subset \Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w))$.

Conversely, for any $x \in \Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w))$, there exists $y \in Y$ such that $(x, y) \in \mathrm{Gr}(F) \cap (\Gamma \times V_w)$. Hence $x \in \Gamma$ and $y \in F(x) \cap V_w$. By (4.2), $x \in S_w$. Hence $\Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w)) \subset S_w$. Therefore (4.3) holds.

In 1953, Arrow, Barankin and Blackwell [1] studied the vector optimization problem (4.1) under some restrictive assumptions. In particular, they established the following celebrated theorem.

Theorem A. Let $X = R^m$, $Y = R^n$ and $C = R^n_+$. Let $T : X \to Y$ be a bounded linear operator and $b \in Y$. Suppose that the objective mapping

$F(x) = \{T(x) + b\}$ for all $x \in X$. Then, the following statements hold.

(i) S_w and V_w are the union of finitely many polyhedra of X if S_w is nonempty.

(ii) S and V are the union of finitely many polyhedra of Y if S is nonempty.

(iii) S_w, V_w, S and V are pathwise connected if S is nonempty.

The following lemma, which are taken from [22] and [21], is a characterization of the set of weak Pareto efficient points.

Lemma 4.2.1. *Let A be a nonempty closed subset of Y. Then*

$$WE(A, C) = A \cap \mathrm{bd}(A + C) = A \backslash (A + \mathrm{int}(C)). \tag{4.4}$$

Proof. Pick $e \in \mathrm{int}(C)$. Let $y \in WE(A, C)$. Then $y - \varepsilon e \notin A + C$ for all $\varepsilon > 0$ (since $\varepsilon + C \subset \mathrm{int}(C)$), so $y \notin \mathrm{int}(A + C)$. Since $y \in A \subset A + C$, it follows that $y \in A \cap \mathrm{bd}(A + C)$. Thus, $WE(A, C) \subset A \cap \mathrm{bd}(A + C)$. The inclusion $A \cap \mathrm{bd}(A+C) \subset A \backslash (A + \mathrm{int}(C))$ follows from the fact that $A + \mathrm{int}(C) \subset \mathrm{int}(A+C)$. Finally, the equivalence of $WE(A, C) = A \backslash (A + \mathrm{int}(C))$ is evident from the definition of weak efficient points. □

Let X, Y be normed spaces and let $C \subset$ be a convex cone. For a set-valued mapping $\Phi : X \rightrightarrows Y$, we denote by $\mathrm{Gr}(\Phi)$ and $\mathrm{epi}_C(\Phi)$ the graph and $C-$epigraph of Φ, respectively, that is,

$$\mathrm{Gr}(\Phi) := \{(x, y) : x \in X \text{ and } y \in \Phi(x)\}$$

and

$$\mathrm{epi}_C(\Phi) := \{(x, y) : x \in X \text{ and } y \in \Phi(x) + C\}.$$

We say that Φ is $C-$convex if $\mathrm{epi}_C(\Phi)$ is convex. Note that Φ is $C-$convex if and only if

$$t\Phi(x_1) + (1-t)\Phi(x_2) \subset \Phi(tx_1 + (1-t)x_2) + C \ \forall x_1, x_2 \in X \text{ and } \forall t \in [0, 1]. \tag{4.5}$$

Indeed, suppose that $\mathrm{epi}_C(\Phi)$ is convex. Take $x_1, x_2 \in X$, $t \in [0, 1]$ and let $y_1 \in \Phi(x_1)$ and $y_2 \in \Phi(x_2)$. Since $\mathrm{epi}_C(\Phi)$ is convex, one has $t(x_1, y_1) + (1 -$

$t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in \mathrm{epi}_C(\Phi)$. Hence, $ty_1 + (1-t)y_2) \in \Phi(tx_1 + (1-t)x_2) + C$, and so (4.5) holds.

Conversely, suppose that (4.5) holds. For any $t \in [0,1]$ and (x_1, y_1+c_1), $(x_2, y_2+c_2) \in \mathrm{epi}_C(\Phi)$ with $y_1 \in \Phi(x_1)$, $y_2 \in \Phi(x_2)$ and $c_1, c_2 \in C$, (4.5) implies that $ty_1 + (1-t)y_2 \in \Phi(tx_1 + (1-t)x_2) + C$, and so

$$t(y_1 + c_1) + (1-t)(y_2 + c_2) \in \Phi(tx_1 + (1-t)x_2) + C.$$

Thus, $t(x_1, y_1 + c_1) + (1-t)(x_2, y_2 + c_2) \in \mathrm{epi}_C(\Phi)$, which means that Φ is C−convex.

In the remainder of this chapter, we always assume that the graph of the objective mapping F in (4.1) is the union of finitely many polyhedra in $X \times Y$.

4.2.1 The general ordering cone case

In this subsection, we consider the optimization problem (4.1) under the assumption that C is a general closed convex cone with nonempty interior and F is C−convex.

For $y^* \in Y^*$ and $A \subset Y$, let

$$\lambda_{y^*}(A) := \inf_{a \in A} \langle y^*, a\rangle \quad \text{and} \quad L_{y^*}(A) := \{a \in A : \langle y^*, a\rangle = \lambda_{y^*}\}. \tag{4.6}$$

The following lemmas are useful for us.

Lemma 4.2.2. *[21, Lemma 3.2] Let Y be a normed space, A be the union of finitely many polyhedra of Y and let Λ be a subset of Y^*. Then, $\cup_{y^* \in \Lambda} L_{y^*}(A)$ is the union of finitely many polyhedra of Y and, more precisely, there exists a finite subset Λ_0 of Λ such that*

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A) = \bigcup_{y^* \in \Lambda_0} L_{y^*}(A). \tag{4.7}$$

Proof. Take finitely many polyhedra $P_1, \cdots, P_n$ of Y such that $A = \cup_{i=1}^n P_i$. Then

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A) = \bigcup_{i=1}^{n} (P_i \cap \bigcup_{y^* \in \Lambda} L_{y^*}(A)) = \bigcup_{i=1}^{n} (\bigcup_{y^* \in \Lambda} P_i \cap L_{y^*}(A)) \tag{4.8}$$

By (4.6), it is easy to verify that $P_i \cap L_{y^*}(A)$ is a face of P_i whenever $P_i \cap L_{y^*}(A)$ is nonempty. That is $\{P_i \cap L_{y^*}(A) : y^* \in \Lambda\}\backslash\{\emptyset\}$ is a family of faces of P_i and hence must be a finite family because P_i is a polyhedron and so has only finitely many faces (see [2]). Therefore there exists a finite subset Λ_i of Λ such that

$$\{P_i \cap L_{y^*}(A) : y^* \in \Lambda\}\backslash\{\emptyset\} = \{P_i \cap L_{y^*}(A) : y^* \in \Lambda_i\},$$

and it follows from (4.8) that

$$\bigcup_{y^*\in\Lambda} L_{y^*}(A) = \bigcup_{i=1}^{n}(\bigcup_{y^*\in\Lambda} P_i \cap L_{y^*}(A)) = \bigcup_{i=1}^{n}(\bigcup_{y^*\in\Lambda_i} P_i \cap L_{y^*}(A)) \subset \bigcup_{i=1}^{n}(\bigcup_{y^*\in\Lambda_i} L_{y^*}(A)).$$

Let $\Lambda_0 = \cup_{i=1}^{n}\Lambda_i$, we see that $\bigcup_{y^*\in\Lambda} L_{y^*}(A) \subset \bigcup_{y^*\in\Lambda_0} L_{y^*}(A)$ and (4.7) is shown as the reverse inclusion is trivial. □

Lemma 4.2.3. *Let Z be a closed subset of a normed space Y. Suppose that $C \subset Y$ is a closed convex cone with nonempty interior and that $Z + C$ is convex. Then,*

$$WE(Z + C, C) = \bigcup_{y^*\in C^+\backslash\{0\}} L_{y^*}(Z + C). \tag{4.9}$$

Consequently,

$$WE(Z, C) = \bigcup_{y^*\in C^+\backslash\{0\}} L_{y^*}(Z). \tag{4.10}$$

Proof. For any $y + c \in WE(Z + C, C)$ with $y \in Z$ and $c \in C$,

$$(y + c - (Z + C)) \cap \text{int}(C) = \emptyset.$$

Since $Z+C$ and $\text{int}(C)$ are convex nonempty sets, the Separation Theorem implies that there exist $y^* \in Y^*\backslash\{0\}$ such that

$$\langle y^*, y + c - z - c'\rangle \leq \langle y^*, c''\rangle \text{ for any } z \in Z, c' \in C \text{ and } c'' \in \text{int}(C).$$

Since $\overline{\text{int(C)}} = C$ and C is a cone, it follows that

$$\langle y^*, y + c - z - c'\rangle \leq \inf_{c''\in C}\langle y^*, c''\rangle = 0 \text{ for any } z \in Z, c' \in C$$

and so $y^* \in C^+\backslash\{0\}$. Thus $y + c \in L_{y^*}(Z + C)$.

Conversely, let $y^* \in C^+\backslash\{0\}$ and $y \in L_{y^*}(Z + C)$. Let $c \in \text{int}(C)$. Then there exists $\varepsilon > 0$ such that $c + \varepsilon B_Y \subset C$, where B_Y denotes the unit open ball in Y. Since $y^* \neq 0$, there exists $b \in B$ such that $\langle y^*, b\rangle < 0$. It follows from $c + \varepsilon b \in C$ and $y^* \in C^+$ that $\langle y^*, c\rangle > 0$. Since $y \in L_{y^*}(Z+C)$, one has $\langle y^*, y - z - c\rangle \leq 0$ for all $z \in Z$ and $c \in C$. Hence, $(y - Z - C) \cap \text{int}(C) = \emptyset$ and so $y \in WE(Z + C, C)$. Thus (4.9) holds. □

Theorem 4.2.4. *[21, Theorem 3.1] Let X, Y be normed spaces and let the ordering cone C have nonempty interior. Suppose that* $\text{Gr}(F)$ *is the union of finitely many polyhedra and that $F(\Gamma) + C$ is convex. Then, V_w is the union of finitely many polyhedra of Y and, more precisely, there exist $y_i^* \in C^+$ with $\|y_i^*\| = 1\,(1 \leq i \leq q)$ such that*

$$V_w = \bigcup_{i=1}^{q} L_{y_i^*}(F(\Gamma)). \tag{4.11}$$

Moreover, S_w is also the union of finitely many polyhedra of X.

Proof. The convexity of $F(\Gamma) + C$ and Lemma 4.2.3 imply that

$$WE(F(\Gamma) + C, C) = \bigcup_{y^* \in C^+\backslash\{0\}} L_{y^*}(F(\Gamma) + C). \tag{4.12}$$

It follows from $WE(F(\Gamma), C) = F(\Gamma) \cap WE(F(\Gamma) + C, C)$ and (4.6) that

$$WE(F(\Gamma), C) = \bigcup_{y^* \in C^+\backslash\{0\}} F(\Gamma) \cap L_{y^*}(F(\Gamma) + C) = \bigcup_{y^* \in C^+\backslash\{0\}} L_{y^*}(F(\Gamma)). \tag{4.13}$$

Since the feasible set Γ is a polyhedron of X and $\text{Gr}(F)$ is the union of finitely many polyhedra of $X \times Y$, $\text{Gr}(F) \cap (\Gamma \times Y)$ is the union of finitely many polyhedra. Noting that $F(\Gamma) = \Pi_Y(\text{Gr}(F) \cap (\Gamma \times Y))$, Corollary 2.3.10 implies that $F(\Gamma)$ is the union of finitely many polyhedra of Y. Noting that $V_w = WE(F(\Gamma), C)$, it follows from (4.13) and Lemma 4.2.2 that V_w is the union of finitely many polyhedra of Y and there exist $y_i^* \in C^+$ with $\|y_i^*\| = 1\,(1 \leq i \leq q)$ such that

(4.11) holds. This and Corollary 2.3.10 imply that $F^{-1}(V_w)$ is the union of finitely many polyhedra. Hence, $S_w = \Gamma \cap F^{-1}(V_w)$ is the union of finitely many polyhedra of X. $\square$

Similar to S_w and V_w, S and V are the union of finitely many polyhedra when the ordering cone C has a weakly compact base.

Theorem 4.2.5. *[21, Theorem 3.2] Let X and Y be normed spaces and let the ordering cone C have a weakly compact base. Suppose that $F(\Gamma) + C$ is convex. Then, S and V are the union of finitely many polyhedra of X and Y, respectively, and, more precisely, there exist $y_i^* \in C^{+i}$ with $\|y_i^*\| = 1\,(1 \leq i \leq q)$ such that*

$$V = \bigcup_{i=1}^{q} L_{y_i^*}(F(\Gamma)). \tag{4.14}$$

Proof. Since the feasible set Γ is a polyhedron of X and $\mathrm{Gr}(F)$ is the union of finitely many polyhedra of $X \times Y$, $\mathrm{Gr}(F) \cap (\Gamma \times Y)$ is the union of finitely many polyhedra. Noting that $F(\Gamma) = \Pi_Y(\mathrm{Gr}(F) \cap (\Gamma \times Y))$, Corollary 2.3.10 implies that $F(\Gamma)$ is the union of finitely many polyhedra of Y. By Corollary 2.3.7, there exist closed subspaces Y_1, Y_2 of Y such that

$$Y = Y_1 + Y_2,\ Y_1 \cap Y_2 = \{0\} \text{ and } \dim(Y_2) < +\infty$$

and

$$F(\Gamma) = Y_1 + \bigcup_{i=1}^{m} H_i,$$

where $H_i\,(1 \leq i \leq m)$ are polyhedra of Y_2. Hence,

$$\mathrm{co}(F(\Gamma)) = Y_1 + \mathrm{co}(\bigcup_{i=1}^{m} H_i).$$

By Lemma 2.3.1, $\mathrm{co}(\bigcup_{i=1}^{m} H_i)$ is a polyhedron of Y_2. This and corollary 2.3.7 imply that $\mathrm{co}(F(\Gamma))$ is a polyhedron. It follows from theorem 3.3.8 that

$$E(\mathrm{co}(F(\Gamma)), C) \subset \mathrm{cl}(\mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i})). \tag{4.15}$$

Noting that $\mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i}) = \bigcup_{y^* \in C^{+i}} L_{y^*}(\mathrm{co}(F(\Gamma)))$, it follows from Lemma 4.2.2 that $\mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i})$ is the union of finitely many polyhedra and so is closed. Since $\mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i})$ is a subset of $E(\mathrm{co}(F(\Gamma)), C)$, (4.15) implies that

$$E(\mathrm{co}(F(\Gamma)), C) = \mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i}). \tag{4.16}$$

On the other hand, the convexity of $F(\Gamma) + C$ implies that $F(\Gamma) \subset \mathrm{co}(F(\Gamma)) \subset F(\Gamma) + C$. Thus,

$$E(F(\Gamma), C) = E(\mathrm{co}(F(\Gamma)), C) \text{ and } \mathrm{Supp}(F(\Gamma), C^{+i}) = \mathrm{Supp}(\mathrm{co}(F(\Gamma)), C^{+i}).$$

This and (4.16) imply that $E(F(\Gamma), C) = \mathrm{Supp}(F(\Gamma), C^{+i}) = \bigcup_{y^* \in C^{+i}} L_{y^*}(F(\Gamma), C)$. By Lemma 4.2.2, $V = E(F(\Gamma), C)$ is the union of finitely many polyhedra of Y and there exist $y_i^* \in C^+$ with $\|y_i^*\| = 1\,(1 \le i \le q)$ such that (4.14) holds. Since $F^{-1}(L_{y_i^*}(F(\Gamma)) = \Pi_X(\mathrm{Gr}(F) \cap (X \times L_{y_i^*}(F(\Gamma))))$ and $\mathrm{Gr}(F) \cap (X \times L_{y_i^*}(F(\Gamma))$ is the union of finitely many polyhedra of $X \times Y$, it follows from Lemma 2.3.9 that $S = \Gamma \cap F^{-1}(V)$ is the union of finitely many polyhedra of X. □

The following lemma is well known, see [11, Theorem 2.8.2].

Lemma 4.2.6. *A normed space is reflexive if and only if its closed unit ball is weakly compact.*

Noting that a closed convex subset of a normed space is weakly closed, lemma 4.2.6 implies that every bounded closed convex subset of a reflexive space is weakly compact, hence, the following corollary is immediate from theorem 4.2.5.

Corollary 4.2.7. *[21, Corollary 3.1] Let X be normed space and Y be a reflexive space. Let the ordering cone C have a bounded base and suppose that $F(\Gamma) + C$ is convex. Then, S and V are the union of finitely many polyhedra of X and Y, respectively, and, more precisely, there exist $y_i^* \in C^{+i}$ with $\|y_i^*\| = 1\,(1 \le i \le q)$ such that*

$$V = \bigcup_{i=1}^{q} L_{y_i^*}(F(\Gamma)).$$

We notice that in a finite dimensional space, every closed convex and pointed cone has a convex and bounded base. The following corollary is a consequence of Corollary 4.2.7

Corollary 4.2.8. *[21, Corollary 3.2] Let X, Y be normed spaces and let the ordering cone C be pointed. Suppose that Y is finite dimensional and that $F(\Gamma) + C$ is convex. Then, S and V are the union of finitely many polyhedra of X and Y, respectively, and, more precisely, there exist $y_i^* \in C^{+i}$ with $\|y_i^*\| = 1\,(1 \le i \le q)$ such that*

$$V = \bigcup_{i=1}^{q} L_{y_i^*}(F(\Gamma)).$$

Corollary 4.2.9. *[21, Corollary 3.3] Let X, Y be normed spaces and let the ordering cone C have a weakly compact base. Let A be the union of finitely many polyhedra of Y. Suppose that $A + C$ is convex and that $E(A, C)$ is nonempty. Then there exist finitely many polyhedra $E_1, \cdots, E_p$ of Y such that*

$$E(A, C) = \mathrm{Supp}(A, C^{+i}) = \bigcup_{i=1}^{p} E_i. \tag{4.17}$$

Proof. Define the set-valued mapping $F : X \rightrightarrows Y$ by $F(x) = A$ for all $x \in X$. Let the feasible set $\Gamma = X$. Then, $\mathrm{Gr}(F) = X \times A$ is the union of finitely many polyhedra of $X \times Y$ and $V = E(A, C)$. By Theorem 4.2.5, there exist $y_i^* \in C^{+i}\,(1 \le i \le p)$ such that $E(A, C) = \bigcup_{i=1}^{p} L_{y_i^*}(A)$. Since $\mathrm{Supp}(A, C^{+i}) \subset E(A, C)$ and $\bigcup_{i=1}^{p} L_{y_i^*}(A) \subset \mathrm{Supp}(A, C^{+i})$, (4.17) holds with $E_i = L_{y_i^*}(A)$. $\square$

In the proofs of Theorems 4.2.4 and 4.2.5, the convexity of $F(\Gamma) + C$ plays an key role. The following example shows that Theorems 4.2.4 and 4.2.5 are not necessarily true without the convexity assumption on $F(\Gamma) + C$.

Example 4.2.1. *[21] Let $X = \mathbb{R}^2, Y = \mathbb{R}^3$ and*

$$C = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : (t_1^2 + t_2^2)^{\frac{1}{2}} \le 2t_3\}.$$

Let $\Gamma = \mathbb{R}^2$ *and define the set-valued mapping* $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ *as*

$$\mathrm{Gr}(F) = \{(t_1, t_2, t_1, t_2, t) \in \mathbb{R}^5 : |t_2| \le t_1 \,\mathrm{and}\, t = \min\{t_1, 1\}\}.$$

Let

$$A_1 := \{(t_1, t_2) \in \mathbb{R}^2 : |t_2| \le t_1 \,\mathrm{and}\, t_1^2 + t_2^2 \ge 4\}$$

and

$$A_2 := \{(t_1, t_2) \in \mathbb{R}^2 : |t_2| \le t_1 \,\mathrm{and}\, t_1^2 + t_2^2 > 4\}.$$

Then,

$$S_w = \{(0,0)\} \cup A_1, \;\; V_w = \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_1\},$$

$$S = \{(0,0)\} \cup A_2, \;\text{ and } V = \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_2\}.$$

Proof. By the definition of F, it is clear that $\mathrm{Gr}(F)$ is the union of finitely many polyhedra of $\mathbb{R}^5$ and

$$F(\Gamma) = \{(t_1, t_2, t) \in \mathbb{R}^3 : |t_2| \le t_1 \,\mathrm{and}\, t = \min\{t_1, 1\}\}. \tag{4.18}$$

Note that

$$S_w = F^{-1}(V_w),\, S = F^{-1}(V),\, V_w = WE(F(\Gamma), C) \text{ and } V = E(F(\Gamma), C).$$

It is sufficient to show that

$$WE(F(\Gamma), C) = \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_1\} \tag{4.19}$$

and

$$E(F(\Gamma), C) = \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_2\}. \tag{4.20}$$

Let $(t_1, t_2, t) \in F(\Gamma)$ but $(t_1, t_2, t) \notin \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_1\}$, that is, $t_1 > 0$ and $t_1^2 + t_2^2 < 4$. Then, (4.18) implies that $(t_1^2 + t_2^2)^{\frac{1}{2}} < 2t$. Hence $(t_1, t_2, t) \in \mathrm{int}(C)$. Thus, $(0,0,0) \in ((t_1, t_2, t) - \mathrm{int}(C)) \cap F(\Gamma)$, and so $(t_1, t_2, t) \notin WE(F(\Gamma), C)$. Therefore,

$$WE(F(\Gamma), C) \subset \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_1\} \tag{4.21}$$

Similarly, let $(t_1, t_2, t) \in F(\Gamma)$ but $(t_1, t_2, t) \notin \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_2\}$, that is, $t_1 > 0$ and $t_1^2 + t_2^2 \leq 4$. Then, (4.18) implies that $(0,0,0) \in (t_1, t_2, t) - C$, and so $(t_1, t_2, t) \notin E(F(\Gamma), C)$. Hence,

$$E(F(\Gamma), C) \subset \{(0,0,0)\} \cup \{(t_1, t_2, 1) : (t_1, t_2) \in A_2\}. \tag{4.22}$$

Noting that $(0,0,0) \in E(F(\Gamma), C) \subset WE(F(\Gamma), C)$, by (4.21) and (4.22), to prove (4.19) and (4.20), we need show that

$$\{(t_1, t_2, 1) : (t_1, t_2) \in A_1\} \subset WE(F(\Gamma), C) \tag{4.23}$$

and

$$\{(t_1, t_2, 1) : (t_1, t_2) \in A_2\} \subset E(F(\Gamma), C). \tag{4.24}$$

To prove (4.23), suppose to the contrary that there exists $(t_1, t_2) \in A_1$ such that $(t_1, t_2, 1) \notin WE(F(\Gamma), C)$. Then, there exists $(\bar{t_1}, \bar{t_2}, \bar{t}) \in F(\Gamma)$ such that

$$(t_1, t_2, 1) - (\bar{t_1}, \bar{t_2}, \bar{t}) \in \mathrm{int}(C), \tag{4.25}$$

that is, $((t_1 - \bar{t_1})^2 + (t_2 - \bar{t_2})^2)^{\frac{1}{2}} < 2(1 - \bar{t})$. This and (4.18) imply that $\bar{t} < 1$, $|\bar{t_2}| \leq \bar{t_1}$ and $\bar{t} = \bar{t_1}$. Hence, $(\bar{t_1}^2 + \bar{t_2}^2)^{\frac{1}{2}} \leq 2\bar{t}$ and so $(\bar{t_1}, \bar{t_2}, \bar{t}) \in C$. This and (4.25) imply that $(t_1, t_2, 1) \in \mathrm{int}(C) + C = \mathrm{int}(C)$ and so $t_1^2 + t_2^2 < 4$, contradicting $(t_1, t_2) \in A_1$. Thus, (4.23) holds.

To prove (4.24), suppose to the contrary that there exist $(t_1, t_2) \in A_2$ and $(t_1', t_2', t) \in F(\Gamma)$ such that $(t_1, t_2, 1) - (t_1', t_2', t) \in C$ and $(t_1, t_2, 1) \neq (t_1', t_2', t)$. Then,

$$((t_1 - t_1')^2 + (t_2 - t_2')^2)^{\frac{1}{2}} \leq 2(1 - t),\ |t_2'| \leq t_1' \text{ and } t = \min\{t_1', 1\}.$$

This and $(t_1, t_2, 1) \neq (t_1', t_2', t)$ imply that $t = t_1' < 1$. Noting that $|t_2'| \leq t_1'$, it follows that $(t_1', t_2', t) \in C$. Noting that $(t_1, t_2, 1) - (t_1', t_2', t) \in C$, it follows that $(t_1, t_2, 1) \in C$, that is, $t_1^2 + t_2^2 \leq 4$, contradicting $(t_1, t_2) \in A_2$. Therefore, (4.24) holds.

□

4.2.2 The polyhedral ordering cone case

In this subsection, we assume that the ordering cone C is polyhedral.

Theorem 4.2.10. *[21, Theorem 3.3] Let X, Y be normed spaces and let the ordering cone C be polyhedral and have nonempty interior. Suppose that* $\mathrm{Gr}(F)$ *is the union of finitely many polyhedra of $X \times Y$. Then, S_w and V_w are the union of finitely many polyhedra of X and Y, respectively.*

Proof. Since $\mathrm{Gr}(F)$ is the union of finitely many polyhedra of $X \times Y$, one has that $\mathrm{Gr}(F) \cap (\Gamma \times Y)$ is the union of finitely many polyhedra too. Note that $F(\Gamma) = \Pi_Y(\mathrm{Gr}(F) \cap (\Gamma \times Y))$. By lemma 2.3.9, there exist finitely many polyhedra $P_1, \cdots, P_n$ of Y such that $F(\Gamma) = \cup_{i=1}^n P_i$. It follows from lemma 4.2.1 that

$$\begin{aligned} V_w = WE(F(\Gamma), C) &= F(\Gamma) \backslash (F(\Gamma) + \mathrm{int}(C)) \\ &= \cup_{i=1}^n (P_i \backslash \cup_{j=1}^n (P_j + \mathrm{int}(C))) \\ &= \cup_{i=1}^n (\cap_{j=1}^n (P_i \backslash (P_j + \mathrm{int}(C)))). \end{aligned}$$

Thus, to prove that V_w is the union of finitely many polyhedra of Y, it is sufficient to show that each $P_i \backslash (P_j + \mathrm{int}(C))$ is the union of finitely many polyhedra. Since the ordering cone C is a polyhedron, 2.3.8 implies that $P_j + C$ is a polyhedron. Take $y_1^*, \cdots, y_p^* \in Y^* \backslash \{0\}$ and $t_1, \cdots, t_p \in \mathbb{R}$ such that

$$P_j + C = \{y \in Y : \langle y_k^*, y \rangle \le t_k, k = 1, \cdots, p\}.$$

Then

$$\mathrm{int}(P_j + C) = \{y \in Y : \langle y_k^*, y \rangle < t_k, k = 1, \cdots, p\}.$$

We claim that

$$\mathrm{int}(P_j + C) = P_j + \mathrm{int}(C). \tag{4.26}$$

Granting this, one has

$$P_j + \mathrm{int}(C) = \cap_{k=1}^p \{u \in Y : \langle y_k^*, y \rangle < t_k\}.$$

Hence,

$$\begin{aligned} P_i\backslash(P_j + \mathrm{int}(C)) &= \cup_{k=1}^{p}(P_i\backslash\{y \in Y : \langle y_k^*, y\rangle < t_k\}) \\ &= \cup_{k=1}^{p}(P_i \cap \{y \in Y : \langle -y_k^*, y\rangle \le -t_k\}) \end{aligned}$$

This shows that $P_i\backslash(P_j + \mathrm{int}(C))$ is the union of finitely many polyhedra and so is V_w. By (4.3), $S_w = \Pi_X(\mathrm{Gr}(F) \cap (\Gamma \times V_w))$. This and lemma 2.3.9 imply that S_w is the union of finitely many polyhedra. Now we show that (4.26) holds. Since $P_j + \mathrm{int}(C)$ is open, it is clear that $P_j + \mathrm{int}(C) \subset \mathrm{int}(P_j + C)$. Conversely, suppose to the contrary that there exist $y + c \in \mathrm{int}(P_j + C)$ with $y \in P_j, c \in C$ such that $y + c \notin P_j + \mathrm{int}(C)$. Since $P_j + \mathrm{int}(C)$ is a nonempty convex set, the Separation Theorem implies that there exists $y^* \in Y^*\backslash\{0\}$ such that

$$\langle y^*, y + c\rangle > \langle y^*, z + c'\rangle \text{ for all } z \in P_j \text{ and } c' \in \mathrm{int}(C).$$

It follows that $\langle y^*, c\rangle = \max\{\langle y^*, c'\rangle : c' \in C\} = 0$ and $\langle y^*, y\rangle = \max\{\langle y^*, z\rangle : z \in P_j\}$. Hence, $\langle y^*, y + c\rangle = \max\{\langle y^*, x\rangle : x \in P_j + C\}$, contradicting $y + c \in \mathrm{int}(P_j + C)$. Therefore, (4.26) holds. □

4.3 Connectedness of solution sets and optimal value sets

In this section, let X, Y be normed spaces and let $C \subset Y$ be a closed convex cone with nonempty interior. we present some connectedness results on the weak Pareto solution set, Pareto solution set and Pareto optimal value set of (4.1) under the assumption that F is C−convex. To do this, we need the following notion: let the set-valued mapping $F' : X \rightrightarrows Y$ be such that

$$\mathrm{Gr}(F') = \mathrm{co}(\mathrm{Gr}(F)) \tag{4.27}$$

and consider the following vector optimization problem:

$$C - \min F' \text{ subject to } x \in \Gamma, \tag{4.28}$$

where Γ is the same feasible set as in (4.1). Let S' (resp. S'_w) and V' (resp. V'_w) denote the set of all Pareto (resp. weak Pareto) solutions of (4.28) and the set of all Pareto (resp. weak Pareto) optimal values of (4.28), respectively.

Lemma 4.3.1. *[21, Lemma 4.1] Let X, Y be normed spaces and $C \subset Y$ be closed convex cone with nonempty interior. Suppose that F is $C-$convex. Then, $S'_w = S_w, S' = S$ and $V' = V$.*

Proof. Since F is $C-$convex, one has that $\mathrm{Gr}(\bar{F})$ is a convex set, where $\bar{F}(x) = F(x) + C$ for all $x \in X$. Thus, $\mathrm{Gr}(F) \subset \mathrm{Gr}(F') \subset \mathrm{Gr}(\bar{F})$, that is,

$$F(x) \subset F'(x) \subset F(x) + C \quad \forall x \in X. \tag{4.29}$$

It follows that $\mathrm{dom}(F) = \mathrm{dom}(F')$ and $F(\Gamma) \subset F'(\Gamma) \subset F(\Gamma) + C$. Hence, $E(F(\Gamma), C) = E(F'(\Gamma), C)$, that is, $V = V'$. By the definition of F', one has $F^{-1}(V) \subset F'^{-1}(V')$. Thus,

$$S = \Gamma \cap F^{-1}(V) \subset \Gamma \cap F'^{-1}(V') = S'.$$

On the other hand, let $x \in S'$. Then, $x \in \Gamma$ and $F'(x) \cap E(F(\Gamma), C) \neq \emptyset$. By $(F(x) + C) \cap E(F(\Gamma), C) = F(x) \cap E(F(\Gamma), C)$ and (4.29), one has $F(x) \cap E(F(\Gamma), C) = F'(x) \cap E(F(\Gamma), C) \neq \emptyset$. Hence, $x \in \Gamma \cap F^{-1}(E(F(\Gamma), C))$ and so $x \in S$. This shows that $S' \subset S$. Therefore, $S' = S$.

Now we prove that $S'_w = S_w$. Let $x \in S'_w$. Then, $x \in \Gamma$ and $F'(x) \cap WE(F'(\Gamma), C) \neq \emptyset$. By (4.29), there exist $y \in F(x)$ and $c \in C$ such that $y + c \in WE(F'(\Gamma), C)$ and so $F'(\Gamma) \cap (y + c - \mathrm{int}(C)) = \emptyset$. Noting that $\mathrm{int}(C) = C + \mathrm{int}(C)$ one has $F'(\Gamma) \cap (y + c - \mathrm{int}(C) - C) = \emptyset$. Hence $(F'(\Gamma) + C) \cap (y + c - \mathrm{int}(C)) = \emptyset$. Since $F(\Gamma) + c \subset F'(\Gamma) + C$, it follows that $F(\Gamma) \cap (y - \mathrm{int}(C) = \emptyset$, that is $y \in WE(F(\Gamma), C)$. So $x \in S_w$ and $S'_w \subset S_w$. Conversely, let $x \in S_w$. Then, there exists $y \in F(x)$ such that $y \in WE(F(\Gamma), C)$. Hence $F(\Gamma) \cap (y - \mathrm{int}(C) = \emptyset$. This and $C + \mathrm{int}(C) = \mathrm{int}(C)$ imply that $(F(\Gamma) + C) \cap (y - \mathrm{int}(C) = \emptyset$. It follows from (4.29) that $F'(\Gamma) \cap (y - \mathrm{int}(C)) = \emptyset$ and so $y \in WE(F'(\Gamma), C)$. So $x \in S'_w$. Therefore $S_w = S'_w$. $\square$

To present the main results of this section, we need the following lemma.

Lemma 4.3.2. *[21, Lemma 4.2] Let Y be a normed space and A be a polyhedron in Y. Let Λ be a convex subset of Y^*. Then, $\bigcup_{y^* \in \Lambda} L_{y^*}(A)$ is pathwise connected.*

Proof. By Corollary 2.3.7, there exist two closed subspaces Y_1 and Y_2 of Y and a polyhedron A_2 in Y_2 such that

$$Y = Y_1 + Y_2, Y_1 \cap Y_2 = \{0\}, \dim(Y_2) < +\infty \text{ and } A = Y_1 + A_2. \tag{4.30}$$

Let y^* in Λ such that $L_{y^*}(A) \neq \emptyset$. The definition of $L_{y^*}(A)$ and (4.30) imply that

$$\langle y^*, y_1 \rangle = 0 \text{ for all } y_1 \in Y_1 \text{ and } L_{y^*}(A) = Y_1 + L_{y^*}(A_2).$$

Hence $\bigcup_{y^* \in \Lambda} L_{y^*}(A) = Y_1 + \bigcup_{y^* \in \Lambda} L_{y^*}(A_2)$. So it is sufficient to show that $\bigcup_{y^* \in \Lambda} L_{y^*}(A_2)$ is pathwise connected. By Lemma 4.2.2, there exist $y_1^*, \cdots, y_n^* \in \Lambda$ such that

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A_2) = \bigcup_{i=1}^{n} L_{y_i^*}(A_2). \tag{4.31}$$

Without loss of generality, we assume that each $L_{y_i^*}(A_2)$ is nonempty. Let $T : Y_2 \to \mathbb{R}^n$ be defined by

$$T(y) := (\langle y_1^*, y \rangle, \cdots, \langle y_n^*, y \rangle) \text{ for all } y \in Y_2.$$

Then, T is a bounded linear operator. Let S_w^T denote the set of all weak Pareto solutions of the following vector optimization problem:

$$\mathbb{R}_+^n - \min T(y) \text{ subject to } y \in A_2.$$

We claim that

$$S_w^T = \bigcup_{y^* \in \Lambda} L_{y^*}(A_2). \tag{4.32}$$

Granting this, (iii) of Theorem A (see page 45) implies that $\bigcup_{y^* \in \Lambda} L_{y^*}(A_2)$ is pathwise connected. Now we show that (4.32) holds. Let $y \in S_w^T$. Then, $T(y) \in WE(T(A_2), \mathbb{R}_+^n)$. By lemma 4.2.3, there exists $(t_1, \cdots, t_n) \in \mathbb{R}_+^n \backslash \{0\}$ such that

$$T(y) \in L_{(t_1, \cdots, t_n)}(T(A_2)),$$

that is,

$$\sum_{i=1}^{n} t_i\langle y_i^*, y\rangle = \min\{\sum_{i=1}^{n} t_i\langle y_i^*, z\rangle : z \in A_2\}.$$

Let $v^* := \frac{\sum_{i=1}^{n} t_i y_i^*}{\sum_{i=1}^{n} t_i}$. The convexity of Λ implies that $v^* \in \Lambda$. Hence $y \in L_{v^*}(A_2)$ and so $y \in \cup_{y^*\in\Lambda} L_{y^*}(A_2)$. Therefore $S_w^T \subset \bigcup_{y^*\in\Lambda} L_{y^*}(A_2)$. Conversely, let $y \in \cup_{y^*\in\Lambda} L_{y^*}(A_2)$. By (4.31), $y \in L_{y_i^*}(A_2)$ for some integer $i \in [1, n]$. Hence $\langle y_i^*, y\rangle = \min_{x\in A_2}\langle y_i^*, x\rangle$ and so $\langle y_i^*, y\rangle - \varepsilon \notin \{\langle y_i^*, x\rangle : x \in A_2\}$ for all $\varepsilon > 0$. Thus, $(T(y) - \mathrm{int}(\mathbb{R}_+^n)) \cap T(A_2) = \emptyset$. Hence $T(y) \in WE(T(A_2), \mathbb{R}_+^n)$ and so $y \in S_w^T$. So $\bigcup_{y^*\in\Lambda} L_{y^*}(A_2) \subset S_w^T$. Therefore, (4.34) holds. □

Theorem 4.3.3. *[21, Theorem 4.1] Let X, Y be normed spaces and let the ordering cone C have a weakly compact base. Suppose that the set-valued objective mapping F is $C-$convex. Then, S and V are pathwise connected.*

Proof. Let $F' : X \rightrightarrows Y$ be the mapping defined by (4.27). Noting that in a finite dimensional space, the convex hull of the union of finitely many polyhedra is a polyhedron. Lemma 2.3.1 and Theorem 2.3.6 imply that $\mathrm{Gr}(F')$ is a polyhedron of $X \times Y$. It follows from Corollary 2.3.10 that $F(\Gamma)$ is a polyhedron of Y. By Corollary 4.2.9,

$$V' = E(F(\Gamma), C) = \mathrm{Supp}(F(\Gamma), C^{+i}) = \bigcup_{y^*\in C^{+i}} L_{y^*}(F'(\Gamma)).$$

It follows from (4.6) that

$$\mathrm{Gr}(F') \cap (\Gamma \times V') = \bigcup_{y^*\in C^{+i}} L_{(0,y^*)}(\mathrm{Gr}(F') \times (\Gamma \times Y)).$$

This and Lemma 4.2.2 imply that $\mathrm{Gr}(F') \cap (\Gamma \times V')$ is pathwise connected. Hence $S' = \Pi_X(\mathrm{Gr}(F') \cap (\Gamma \times V'))$ and $V' = \Pi_Y(\mathrm{Gr}(F') \cap (\Gamma \times V'))$ are pathwise connected. It follows from Lemma 4.3.1 that S and V are pathwise connected. □

Since every closed convex pointed cone in a finite dimensional dpace has a weakly compact base, the following corollary is immediate from Theorem 4.3.3.

Corollary 4.3.4. *[21, Corollary 4.1] Let X, Y be normed spaces with* $\dim(Y) < +\infty$ *and let the ordering cone C be pointed. Let A be the union of finitely many polyhedra of Y. Suppose that $A + C$ is convex. Then, $E(A, C)$ is pathwise connected.*

Theorem 4.3.5. *[21, Theorem 4.2] Let X, Y be normed spaces and let the ordering cone C have nonempty interior. Suppose that the objective set-valued mapping F is $C-$convex. Then S_w is pathwise connected.*

Proof. Let F' be as in the proof of Theorem 4.3.3. Then, $\mathrm{Gr}(F')$ is a polyhedron of $X \times Y$ and $F'(\Gamma)$ is a polyhedron of Y. We claim that

$$V'_w = \bigcup_{y^* \in C^+ \setminus \{0\}} L_{y^*}(F'(\Gamma)). \tag{4.33}$$

Granting this, one has

$$\mathrm{Gr}(F') \cap (\Gamma \times V'_w) = \bigcup_{y^* \in C^+ \setminus \{0\}} L_{(0,y^*)}(\mathrm{Gr}(F') \cap (\Gamma \times Y)).$$

It follows from Lemma 4.3.2 that $\mathrm{Gr}(F') \cap (\Gamma \times V'_w)$ is pathwise connected. Hence $S'_w = \Pi_X(\mathrm{Gr}(F') \cap (\Gamma \times V'_w))$ is pathwise connected. By Lemma 4.3.1, $S_w = S'_w$ is pathwise connected. Now we show that (4.33) holds. Let $y^* \in C^+ \setminus \{0\}$ and $y \in L_{y^*}(F'(\Gamma))$. Then,

$$(y - \mathrm{int}(C)) \cap F'(\Gamma) = \emptyset.$$

This and (4.6) imply that $(y - \mathrm{int}(C)) \cap F'(\Gamma) = \emptyset$, and so $y \in WE(F(\Gamma), C) = V'_w$.

Conversely, let $y \in V'_w = WE(F(\Gamma), C)$. Then, $(y - \mathrm{int}(C)) \cap F'(\Gamma) = \emptyset$. Noting that $y - \mathrm{int}(C)$ and $F'(\Gamma)$ are convex, by the Separation Theorem, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y^*, y - c' \rangle > \langle y^*, z \rangle \text{ for any } c' \in \mathrm{int}(C) \text{ and } z \in F'(\Gamma).$$

This implies that $y^* \in C^+ \setminus \{0\}$ and $y \in L_{y^*}(F'(\Gamma))$. Hence $V'_w \subset \bigcup_{y^* \in C^+ \setminus \{0\}} L_{y^*}(F'(\Gamma))$ and (4.33) holds. □

4.4 Optimality conditions of piecewise linear mappings

In this subsection, let X, Y be normed spaces and let $C \subset Y$ be a closed convex pointed cone with nonempty interior, which specifies a preorder $\leq_C$ in Y: for $y_1, y_2 \in Y$, $y_1 \leq_C y_2 \Leftrightarrow y_2 - y_1 \in C$. By $y_1 <_C y_2$, we mean that $y_2 - y_1 \in \text{int}(C)$. If additional conditions are imposed, they will be explicitly specified.

Recall that a mapping $f : X \to Y$ is said to be piecewise linear if there exist polyhedra $P_1, \cdots, P_m$ in X, $\{T_1, \cdots, T_m\} \subset L(X,Y)$ and $\{b_1. \cdots, b_m\} \subset Y$ such that

$$X = \bigcup_{i=1}^{m} P_i \text{ and } f(x) = T_i(x) + b_i \ \forall x \in P_i \,\text{and}\, 1 \leq i \leq m. \tag{4.34}$$

Let $a_1^*, \cdots, , a_n^* \in X^*$ and $r_1, \cdots, r_n \in \mathbb{R}$. Considering the following multiobjective (not necessary convex) piecewise linear optimization problem:

$$C - \min f(x) \text{ subject to } x \in \Gamma, \tag{4.35}$$

where the feasible set $\Gamma := \{x \in X : \langle a_j^*, x\rangle \leq r_j, j = 1, \cdots, n\}$.

A vector $\bar{x} \in \Gamma$ is called a weak Pareto solution of (4.35) if $f(\bar{x}) \in WE(f(\Gamma), C)$; in this case, $f(\bar{x})$ is called a weak Pareto value of (4.35). We denote by S_w and V_w the set of all weak Pareto solutions (4.35) and the set of all weak Pareto values of (4.35), respectively. It is clear that

$$S_w = \Gamma \cap f^{-1}(V_w) \text{ and } V_w = WE(f(\Gamma), C).$$

In sections 4.2 and 4.3, we presented some results on the structure of S_w if $S_w \neq \emptyset$. In this section, we study the optimality conditions of (4.35) and present a condition such that $S_w \neq \emptyset$. We introduce some well-known definitions and properties in convex analysis, see [4] and [18] for instance.

Let f be a convex function from a normed space X to $\mathbb{R} \cup \{+\infty\}$. The domain of f is defined by

$$\text{dom}(f) := \{x \in X : f(x) < +\infty\}.$$

We say that f is proper if $\mathrm{dom}(f) \neq \emptyset$. For a convex proper functional f, the subdifferential of f at $x_0 \in \mathrm{dom}(f)$ is defined by

$$\partial f(x_0) := \{x^* \in X^* : \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) \;\; \forall x \in \mathrm{dom}(f)\}.$$

We denote $\partial f(x) = \emptyset$ if $x \notin \mathrm{dom}(f)$.

Remark 4.4.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex, $x^* \in X^*$ and $x_0 \in \mathrm{dom}(f)$. If there exists $r > 0$ such that,*

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) \;\; \forall x \in \mathrm{dom}(f) \cap B(x_0, r), \tag{4.36}$$

then $x^* \in \partial f(x_0)$, where $B(x_0, r)$ denotes the open ball of X with center x_0 and radius r. Consequently, two convex functions have the same subdifferential at a given point if they have the same value in a neighborhood of that point.

Indeed, let $x^* \in X^*$ such that (4.36) holds. For any $y \in \mathrm{dom}(f)$, there exists $\lambda \in (0,1)$ such that $\lambda x_0 + (1-\lambda)y \in B(x_0, r) \cap \mathrm{dom}(f)$. Hence

$$\begin{aligned}\langle x^*, \lambda x_0 + (1-\lambda)y - x_0 \rangle &\leq f(\lambda x_0 + (1-\lambda)y) - f(x_0) \\ &\leq \lambda f(x_0) + (1-\lambda) f(y) - f(x_0).\end{aligned}$$

Since $1 - \lambda > 0$, divide $1 - \lambda$ on both sides of above inequality,

$$\langle x^*, y - x_0 \rangle \leq f(y) - f(x_0).$$

Thus, $x^* \in \partial f(x_0)$.

For a closed convex subset A of a normed space X, the normal cone of A at $a \in A$ is defined by

$$N(A, a) := \{x^* \in X^* : \langle x^*, x - a \rangle \leq 0 \;\; \forall x \in A\}.$$

Proposition 4.4.1. *[18, Propositions 4.5.1 and 4.5.2] Let $f_i \, (i = 1, \cdots, n)$ be a collection of proper convex functions from a normed space X to $\mathbb{R} \cup \{+\infty\}$. Then the following statements hold.*

(a) If $f(x) = \max\{f_i(x) : i \in \{1, \cdots, n\}\}$ *for all* $x \in X$*, then*

$$\partial f(x) = \overline{\mathrm{co}}^*\Big(\bigcup_{i \in I(x)} \partial f_i(x)\Big), \quad \forall x \in \mathrm{dom}(f)$$

where $\overline{\mathrm{co}}^*$ *denotes the weak star closed convex hull and* $I(x) = \{i \in \{1, \cdots, n\} : f(x) = f_i(x)\}$ *for all* $x \in X$.

(b) If there exists $\bar{x} \in (\mathrm{dom} f_1) \cap (\mathrm{int}\,\mathrm{dom} f_2) \cap \cdots \cap (\mathrm{int}\,\mathrm{dom} f_n)$ *such that* f_i *is continuous at* $\bar{x}$ *for* $i = 2, \cdots, n$*, then*

$$\partial(\sum_{i=1}^{n} f_i)(x) = \sum_{i=1}^{n} \partial f_i(x),$$

for all $x \in \bigcap_{i=1}^{n} \mathrm{dom}(f_i)$.

Proposition 4.4.2. *[19, Theorem 2.4.14 (vii)] Let* X, Y *be normed spaces and* $T \in L(X, Y)$*. Suppose that* $h : Y \to \mathbb{R}$ *is a sublinear and continuous function. Then,* $\partial(h \circ T)(0) = w^* - \mathrm{cl}(T^*(\partial h(0)))$.

Let $I := \{1, \cdots, m\}$ and $I_0 := \{i \in I : \mathrm{int}(P_i) \neq \emptyset\}$, where m is as in (4.35). By Lemma 2.3.11, $X = \bigcup_{i \in I_0} P_i$. Let $\tilde{I}(x) := \{i \in I_0 : x \in P_i\}$ for all $x \in X$.

Lemma 4.4.3. *[22, Lemma 4.1] Let* X, Y *be normed spaces and* f *be defined by* (4.34)*. Suppose that* f *is* $C-$*convex. Let* $\bar{x} \in \Gamma$ *and* $y^* \in C^+$*. Then, the following statements hold.*

(i) $\langle y^*, f(x)\rangle = \max_{i \in \tilde{I}(\bar{x})} \langle y^*, T_i(x) + b_i\rangle$ *for all* $x \in B(\bar{x}, \delta)$ *and some* $\delta > 0$.

(ii) Define $\phi(x) = d(f(x) - f(\bar{x}), -C)$ *for all* $x \in X$*. Then,*

$$\partial \phi(\bar{x}) = \left\{ \sum_{i \in \tilde{I}(\bar{x})} t_i T_i^*(c^*) : \sum_{i \in \tilde{I}(\bar{x})} t_i = 1,\ t_i \geq 0 \text{ and } c^* \in B_{X^*} \cap C^+ \right\},$$

where T_i^* *denotes the conjugate operator of* T_i *and* B_{X^*} *denotes the closed unit ball of* X^*.

Proof. Since each polyhedron is closed, there exists $\delta > 0$ such that $B(\bar{x}, \delta) \cap \bigcup_{i \in I_0 \setminus \bar{I}(\bar{x})} P_i = \emptyset$. This and Lemma 2.3.11 imply that

$$B(\bar{x}, \delta) \subset \bigcup_{i \in \bar{I}(\bar{x})} P_i. \tag{4.37}$$

Hence, for any $x \in B(\bar{x}, \delta)$ there exists $i \in \bar{I}(\bar{x})$ such that $f(x) = T_i(x) + b_i$, and so

$$\langle y^*, f(x) \rangle \leq \max_{i \in \bar{I}()} \langle y^*, T_i(x) + b_i \rangle.$$

It follows from $y^* \in C^+$ and Theorem 2.3.12 that

$$\langle y^*, f(x) \rangle = \max_{i \in \bar{I}(\bar{x})} \langle y^*, T_i(x) + b_i \rangle \ \forall x \in B(\bar{x}, \delta).$$

Hence (i) holds.

To prove (ii), let $i \in \bar{I}(\bar{x})$. Then $f(\bar{x}) = T_i(\bar{x}) + b_i$. For any $x \in X$, noting that f is C−convex, Theorem 2.3.12 implies that $T_i(x) + b_i \leq_C f(x)$ and so $T_i(x - \bar{x}) \leq_C f(x) - f(\bar{x})$. Hence

$$d(T_i(x - \bar{x}), -C) \leq d(T_i(x - \bar{x}), -(C + f(x) - f(\bar{x}) - T_i(x - \bar{x}))) = \phi(x) \text{ for any } x \in X.$$

Let $\psi(x) := \max_{i \in \bar{I}(\bar{x})} d(T_i(x - \bar{x}), -C)$ for all $x \in X$. Then $\psi(x) \leq \phi(x)$ for all $x \in X$. On the other hand, by (4.37), for each $x \in B(\bar{x}, \delta)$, there exists $i \in \bar{I}(\bar{x})$ such that $x \in P_i$ and so $f(x) = T_i(x) + b_i$. Noting that $f(\bar{x}) = T_i(\bar{x}) + b_i$, one has $\phi(x) = d(T_i(x - \bar{x}), -C) \leq \max_{i \in \bar{I}(\bar{x})} d(T_i(x - \bar{x}), -C)$. Therefore, $\phi(x) = \psi(x)$ for all $x \in B(\bar{x}, \delta)$. By Remark 4.4.1, $\partial\phi(\bar{x}) = \partial\psi(\bar{x})$. It follows from Proposition 4.4.1 and 4.4.2 that (ii) holds. $\square$

Let $\bar{x} \in \Gamma$, recall that $\bar{x}$ is a sharp Pareto solution of multiobjective piecewise linear optimizarion problem (4.35) if there exists a constant $\tau \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \leq \tau(d(f(x) - f(\bar{x}), -C) + d(x, \Gamma)) \text{ for all } x \in X. \tag{4.38}$$

It is clear that $\bar{x}$ is a Pareto solution of (4.35) if (4.38) holds.

Indeed, for any $f(y) \in (f(\bar{x}) - C)$ with $y \in \Gamma$, $f(y) - f(\bar{x}) \in -C$. By (4.38), $\|y - \bar{x}\| = 0$ and so $x = y$. Therefore, $(f(\bar{x}) - C) \cap f(\Gamma) = \{f(\bar{x})\}$. This means $\bar{x}$ is a Pareto solution of (4.35).

For any $u \in \Gamma$, let

$$J(u) := \{1 \leq j \leq n : \langle a_j^*, u\rangle = c_j\},$$

where a_j^* and c_j are as in (4.35), and let

$$\Lambda^+(u) := \left\{ \sum_{i\in \bar{I}(u)} t_i T_i^*(y^*) + \sum_{j\in J(u)} s_j a_j^* : y^* \in C^+\backslash\{0\},\, t_i \geq 0,\, s_j \geq 0 \,\text{and} \sum_{i\in\bar{I}(u)} t_i = 1 \right\}.$$

The following theorem is a main result of this section.

Theorem 4.4.4. *[22, Theorem 4.1] Let X, Y be normed spaces, Γ be the feasible set of* (4.35), *and f be defined as* (4.34). *Suppose that f is $C-$convex and $\bar{x} \in \Gamma$. Then the following statements hold.*

(i) $\bar{x}$ is a weak Pareto solution of (4.35) *if and only if $0 \in \Lambda^+(\bar{x})$.*

(ii) $\bar{x}$ is a sharp Pareto solution of (4.35) *if and only if 0 is an interior point of $\Lambda^+(\bar{x})$.*

Proof. By Lemma 4.2.3,

$$V_w = \bigcup_{y^*\in C^+\backslash\{0\}} L_{y^*}(f(\Gamma)).$$

Hence,

$$\bar{x} \in S_w \Leftrightarrow \langle y^*, f(\bar{x})\rangle = \inf_{x\in\Gamma}\langle y^*, f(x)\rangle \text{ for some } y^* \in C^+\backslash\{0\}. \tag{4.39}$$

For each $y^* \in C^+\backslash\{0\}$, let $\phi_{y^*}(x) = \max_{i\in\bar{I}(\bar{x})}\langle y^*, T_i(x) + b_i\rangle$ for all $x \in X$. Then, by (i) of Lemma 4.4.3 and the convexity of f,

$$\phi_{y^*}(\bar{x}) = \inf_{x\in\Gamma} \phi_{y^*}(x) \Leftrightarrow \langle y^*, f(\bar{x})\rangle = \inf_{x\in\Gamma}\langle y^*, f(x)\rangle. \tag{4.40}$$

By Lemma 3.3.1,

$$\phi_{y^*}(\bar{x}) = \inf_{x\in\Gamma} \phi_{y^*}(x) \Leftrightarrow 0 \in \partial\phi_{y^*}(\bar{x}) + N(\Gamma, \bar{x}),$$

where $N(\Gamma, \bar{x})$ denotes the normal cone of Γ at $\bar{x}$ in the sense of convex analysis. Since

$$\partial\phi_{y^*}(\bar{x}) = \left\{ \sum_{i\in\bar{I}(\bar{x})} t_i T_i^*(y^*) : t_i \geq 0, \text{ and } \sum_{i\in\bar{I}(\bar{x})} t_i = 1 \right\} \tag{4.41}$$

and

$$N(\Gamma, \bar{x}) = \left\{ \sum_{j\in J(\bar{x})} s_j x_j^* : s_j \geq 0,\ j \in J(\bar{x}) \right\}, \tag{4.42}$$

by (4.39), (i) holds.

To prove (ii), let

$$h(x) := d(f(x) - f(\bar{x}), -C) + d(x, \Gamma) \text{ for all } x \in X.$$

Suppose that $\bar{x}$ is a sharp Pareto solution of (4.35). Noting that $h(\bar{x}) = 0$ and $\|x - \bar{x}\| \leq \tau h(x)$, one has

$$\langle x^*, x - \bar{x}\rangle \leq \|x - \bar{x}\| \leq \tau(h(x) - h(\bar{x})) \text{ for all } x^* \in B_{X^*}.$$

Hence $\frac{1}{\tau}B_{X^*} \subset \partial h(\bar{x})$. Since $\partial h(\bar{x}) = \partial\phi(\bar{x}) + \partial d(\cdot, \Gamma)(\bar{x})$ and $\partial d(\cdot, \Gamma)(\bar{x}) = B_{X^*} \cap N(\Gamma, \bar{x})$, by (ii) of Lemma 4.4.3, $\partial h(\bar{x}) \subset \Lambda^+(\bar{x})$ and so 0 is an interior point of $\Lambda^+(\bar{x})$. To prove the sufficient part of (ii), suppose that there exists $r > 0$ such that $rB_{X^*} \subset \Lambda^+(\bar{x})$, that is, $rB_{X^*} \subset \partial\phi(\bar{x}) + N(\Gamma, \bar{x})$. By (ii) of Lemma 4.4.3, $\partial\phi(\bar{x})$ is bounded, let $M = \sup\{\|x^*\| : x^* \in \partial\phi(\bar{x})\}$. Hence

$$rB_{X^*} \subset \partial\phi(\bar{x}) + (M + r)B_{X^*} \cap N(\Gamma, \bar{}) \subset (M + r + 1)(\partial\phi(\bar{}) + \partial d(\cdot, \Gamma)(\bar{x})).$$

Let $x \in X$. Then there exist $x^* \in B_{X^*}$, $x_1^* \in \partial\phi(\bar{x})$ and $x_2^* \in \partial d(\cdot, \Gamma)(\bar{x})$ such that $\|x - \bar{x}\| = \langle x^*, x - \bar{x}\rangle$ and $rx^* = (M + r + 1)(x_1^* + x_2^*)$. Hence,

$$\begin{aligned} \frac{r}{M + r + 1}\|x - \bar{x}\| &= \langle x_1^*, x - \bar{x}\rangle + \langle x_2^*, x - \bar{x}\rangle \\ &\leq \phi(x) - \phi(\bar{x}) + d(x, \Gamma) - d(\bar{x}, \Gamma) \\ &= \phi(x) + d(x, \Gamma). \end{aligned}$$

Hence (4.38) holds with $\tau = \frac{M+r+1}{r}$, and so $\bar{x}$ is a sharp Pareto solution of (4.35). Therefore (ii) holds. $\square$

Corollary 4.4.5. *[22, Corollary 4.1] Let X, Y be normed spaces, $T \in L(X,Y)$ and $b \in Y$. Suppose that $f(x) = T(x) + b$ for all $x \in X$. Then, $S_w \neq \emptyset$ if and only if there exist $y^* \in C^+ \backslash \{0\}$ and $s_j \geq 0\,(j = 1, \cdots, n)$ such that*

$$T^*(y^*) + \sum_{j=1}^{n} s_j a_j^* = 0 \tag{4.43}$$

Proof. The necessity is an immediate consequence of Theorem 4.3.3. To prove the sufficiency part, suppose that there exist $y_0^* \in C^+\backslash\{0\}$ and $s_j \geq 0$ such that

$$T^*(y_0^*) + \sum_{j=1}^{n} s_j a_j^* = 0. \tag{4.44}$$

Let $X_0 := \bigcap_{j=1}^{n} \ker(a_j^*)$. Then there exists a finite dimensional subspace X_1 of X such that $X = X_0 + X_1$ and $X_0 \cap X_1 = \{0\}$. Let

$$\Gamma_1 := \{x \in X_1 : \langle a_j^*, x\rangle \leq r_j, j = 1, \cdots, n\}.$$

Then $\Gamma = \Gamma_1 + X_0$. Noting that $\dim(X_1) < +\infty$, Lemma 2.3.1 implies that there exist $e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}$ in X_1 such that

$$\Gamma_1 = \{\Sigma_{i=1}^{p+q} t_i e_i : t_i \geq 0\, i = 1, \cdots, p+q, \text{ and } \Sigma_{i=1}^{p} t_i = 1\}.$$

It is easy to verify that $\langle a_j^*, e_i\rangle \leq 0$ for all $j \in \{1, \cdots, n\}$ and for all $i \in \{p+1, \cdots, p+q\}$. Hence there exists an integer $i_0 \in [1, k]$ such that

$$\sum_{j=1}^{n} \langle s_j a_j^*, e_{i_0}\rangle = \max_{x \in \mathrm{co}(e_1, \cdots, e_p)} \langle \sum_{j=1}^{n} s_j a_j^*, x\rangle = \max_{x \in \Gamma_1} \langle \sum_{j=1}^{n} s_j a_j^*, x\rangle = \max_{x \in \Gamma} \langle \sum_{j=1}^{n} s_j a_j^*, x\rangle$$

This and (4.44) imply that $\langle T^*(y_0^*), e_{i_0}\rangle = \min_{x\in\Gamma}\langle T^*(y_0^*), x\rangle$. Hence, $\langle y_0^*, T(e_{i_0})\rangle = \min_{x\in\Gamma}\langle y_0^*, T(x)\rangle$ and so $\langle y_0^*, f(e_{i_0})\rangle = \min_{x\in\Gamma}\langle y_0^*, f(x)\rangle$. By (4.39), $e_{i_0} \in S_w$ and so $S_w \neq \emptyset$. $\square$

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